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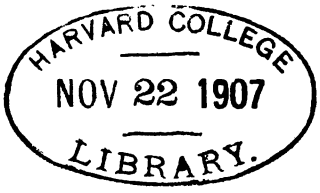

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$$6(y^2+zx)^2+4y(z^3+x^3)=b^4+5y^4 \quad \ldots\ldots(2),$$
$$6(z^2+xy)^2+4z(x^3+y^3)=c^4+5z^4 \quad \ldots\ldots(3)$$

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$$x=(\alpha z+\beta)/(\gamma z+\delta)$$

ramène l'équation

$$a_0x^n+\frac{n}{1}a_1x^{n-1}+\frac{n(n-1)}{1.2}a_2x^{n-2}+\ldots+a_n=0$$

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à la forme binôme $x^n - p = 0$, il faut et il suffit que les déterminants formés avec trois colonnes quelconques du tableau

$$\begin{matrix} a_0 & a_1 & a_2 & \dots & a_{n-2} \\ a_1 & a_2 & a_3 & \dots & a_{n-1} \\ a_2 & a_3 & a_4 & \dots & a_n \end{matrix}$$

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$$\tfrac{1}{2}s, \quad \sqrt{[s(s-b)(s-c)/a]}, \quad 2(s-b)(s-c)/a$$

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$$z = ax^2 + by^2,$$

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$$m^3 - pm^2 + qm - r = 0,$$

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16062. (Robt. W. D. Christie.)—Why is the remainder of

$$100000010000001/109$$

or
$$100000010000001 \pmod{109} = 1 + 10.19^2 + (10.19^2)^2$$
$$= 1 + 13 + 13^2 \pmod{109}$$
$$= 74 \pmod{109}\ ?$$

Generalize for any prime divisor 19

16065. (V. Ramaswami Aiyar.)—Let S denote the parabolic spiral $r = a\theta^2$. Let S′ denote the connected curve $r = a(\theta^2 - 1)$. Let S″ denote the negative pedal of S′. Prove (1) that any arc of the curve S″ is equal in length to the corresponding arc of its pedal S′; (2) that the radius of curvature at any point of S″ is equal to the distance of that point from the origin; (3) that the locus of the point of intersection of normals at corresponding points of S′ and S″ is a spiral of Archimedes; (4) that the evolute of S″ is an involute of a circle; and (5) that any arc of the curve S is equal in length to the corresponding arc of its second negative pedal 35

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$$1 - \frac{C_1}{3} + \frac{C_2}{6} - \frac{C_3}{10} + \frac{C_4}{15} - \ldots$$

where $C_r \equiv$ combinations of n things r together 25, 46

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16083. (Professor Sanjána, M.A.)—Prove that the infinite product

$$\left(1\mp\frac{n^2}{1^2}\right)\left(1\mp\frac{n^2}{2^2}\right)\left(1\mp\frac{n^2}{4^2}\right)\left(1\mp\frac{n^2}{5^2}\right)\left(1\mp\frac{n^2}{7^2}\right)\left(1\mp\frac{n^2}{8^2}\right)\left(1\mp\frac{n^2}{10^2}\right)\ldots$$

$$=\tfrac{1}{3}\left(2\frac{\cos}{\cosh}\tfrac{2}{3}\pi n+1\right) \quad \ldots\ldots \text{ 33}$$

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$$E_m = 2^{2^m}, \quad F_n = E_n + 1, \quad G_n = E_n - 1.$$

Show that every F_n (> 5) may be expressed algebraically in the form

$$2F_n = t^2 - G_n \cdot u^2,$$

and obtain the expression. Discuss the possibility of obtaining a second expression $2F_n = t'^2 - G_n \cdot u'^2$ which, together with the former, shall be suitable for factorizing F_n 21

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16105. (A. M. Nesbitt, M.A.)—A chord of a parabola is such that the normals at its extremities meet on the curve. Show, by pure geometry, that the pole of the chord lies on a certain straight line 82

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16110. (James Blaikie, M.A.)—Trace from geometrical considerations the locus of a point P which moves so that two given straight lines AB, CD subtend the same angle at the point, considering the cases where ABCD is (1) a square, (2) a rectangle, (3) a parallelogram, (4) an irregular quadrilateral 44

16112. (H. Bateman, B.A.)—Prove that (with the usual notation for the Legendre polynomial), if x, y, z lie between ± 1,

$$\sum_0^\infty (2n+1)\, P_n(x)\, P_n(y)\, P_n(z) = \frac{2}{\pi\sqrt{(1-x^2-y^2-z^2+2xyz)}} \quad \text{or} \quad 0,$$

according as the quantity under the square root is positive or negative. 34

16114. (*Suggested in conversation by the late* Professor P. G. Tait. *Communicated by* J. Blaikie, M.A.)—A. and B. play golf, and at each hole it is equally probable that the hole may be won by A., won by B., or halved. Find A.'s chances of winning, halving, or losing the match, if he is (1) 2 up and 3 to play, (2) 2 up and 4 to play, (3) m up and $m+n$ to play 98

16115. (Hon. G. R. Dick.)—Show by pure algebra (without importing any geometrical terms or notions) that the homogeneous equation of the second degree in x, y, z,

$$\begin{vmatrix} x^2, & x_1^2, & x_2^2, & x_3^2, & x_4^2, & a \\ y^2, & y_1^2, & y_2^2, & y_3^2, & y_4^2, & b \\ z^2, & z_1^2, & z_2^2, & z_3^2, & z_4^2, & c \\ yz, & y_1z_1, & y_2z_2, & y_3z_3, & y_4z_4, & f \\ zx, & z_1x_1, & z_2x_2, & z_3x_3, & z_4x_4, & g \\ xy, & x_1y_1, & x_2y_2, & x_3y_3, & x_4y_4, & h \end{vmatrix} = 0$$

(wherein x_1, y_1, z_1, ... are given, and a, ..., h arbitrary constants), which appears to involve five arbitrary parameters, in reality involves only one 53

16116. (H. L. Trachtenberg, B.A.)—If the capitals are the minors of the corresponding small letters in $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, prove that, if

$$F/f = G/g = H/h,$$

then
$$(A/f + H/g + G/h)/(1/f) = (H/f + B/g + F/h)/(1/g) = (G/f + F/g + C/h)/(1/h) \quad \ldots\ldots 43$$

16117. (Lt.-Col. Allan Cunningham, R.E.)—Show that

$$n^{2n} \equiv 1 \pmod{p}$$

where $p = x^4 + y^4 = 256n + 1$ (a prime) and $y = 4\eta$ 42

16119. (R. F. Davis, M.A.)—Eliminate λ, κ from the three equations $(\lambda^2+1)x - \lambda\kappa = a$, $(\lambda^2+1)y + \kappa = \lambda^2 a$, $(\lambda^2+1)(x^2+y^2) = \kappa^2$ (i., ii., iii.) 56

16120. (K. S. Patrachan.)—Show that the diameter of curvature at any point of an ellipse is to the normal chord at the same point in the duplicate ratio of the diameters of the ellipse respectively perpendicular and parallel to the chord 63

16121. (Professor Neuberg.) — On considère les hyperboles équilatères qui passent par deux points fixes A, B et dont un axe de symétrie passe par un point donné C. Trouver le lieu des centres 58

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16122. (A. M. Nesbitt, M.A.) — The tangents to a hyperbola at P and Q meet an asymptote in X and Y. Prove that PQ bisects XY 70

16125. (Professor Nanson.)—If the perpendiculars from the vertices of one tetrahedron upon the faces of a second are concurrent, so are the perpendiculars from the vertices of the second upon the faces of the first 78

16127. (C. E. Youngman, M.A.)—Construct the two circles of which there are given the two centres of similitude, the length of one common tangent, and the direction of another 40

16132. (A. M. Nesbitt, M.A.)—Factorize (A, B, C being angles of a triangle) $\begin{vmatrix} 0, & x\cos C+y, & x\cos B+z \\ y\cos C+x, & 0, & y\cos A+z \\ z\cos B+x, & z\cos A+y, & 0 \end{vmatrix}$ 59

16135. (Lt.-Col. Allan Cunningham, R.E.)—Discuss the possibility of expression of a given number N in one or more of the four forms $(t^2\sim 2u^4)$, $(t^4\sim 2u^2)$ 60

16138. (*Communicated by* A. Thomson.)—If $x_1, x_2, x_3, y_1, y_2, y_3$ be the roots of the cubics in x, y obtained by eliminating y, x respectively between the equations

$$(ax+hy+g)/x = (hx+by+f)/y = gx+fy+c,$$

show that $x_2x_3+y_2y_3 = x_3x_1+y_3y_1 = x_1x_2+y_1y_2 = -1$ 66

16139. (M. V. Arunachalan, M.A.) — If in the expression $\frac{1}{1-x+x^2}$ the sum of the even terms be $\phi(x)$ and the sum of the odd terms be $\psi(x)$, then $[\phi(x)]^2-[\psi(x)]^2 = \phi(-x^2)+\psi(-x^2)$ 52

16144. (W. F. Beard, M.A.)—P, p are corresponding points on an ellipse and its auxiliary circle. Tangents pQ, pQ' are drawn to the ellipse and meet the circle again at q, q'. Prove (1) that QQ', qq', and the tangents at P to the ellipse and at p to the circle all meet on the axis; (2) that, if the poles of QQ', qq' to the circle and of qq' to the ellipse are R, r, and R′, and if pP meets QQ', qq' at L, l, then the loci of L, l, R, R′, r are ellipses concentric with the original ellipse 74

16145. (W. Austin Sleigh, B.A.) — In Question 15909 prove that, if ξ be centre of ellipse, I be in-centre, and I_1 ex-centre, corresponding to A, of the triangle ABC, (1) foot of directrix, K, is at distance $s/2e$ from ξ; (2) BI meets AX at distance $e(s-c)$ from X, and BI_1 meets it at distance $es^2/(3c-b-a)$ from ξ; (3) length of semi-diameter conjugate to ξB is $\sqrt{(r_1r_2)}$; (4) $I_1B : I_1C = \sqrt{r_2} : \sqrt{r_3}$; (5) the parallel straight line to BI_1 through ξ meets BC at distance $(BL=)\,s/2$ from B; (6) $4\xi T.CL = es^2$, T being intersection of BI_1 and AX; (7) ratio of CL to distance of foot of B ordinate from ξ is that of the eccentricity 104

16147. (Professor R. W. Genese, M.A.) — A given mass is distributed over three given points A, B, C. Prove that the moment of inertia of any distribution about its centroid varies as the power of that point with respect to the circle ABC 91

16149. (Robt. W. D. Christie.) — When 12345679 is multiplied by 9 we get a succession of units. Explain the reason and give the general theorem. Put also in Hensel's notation, using prime 7 ... 67

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16150. (Lt.-Col. Allan Cunningham, R.E.)—Given

$$N^x = 549,350,809,785,675,865,253,538,956,347,403,$$

to find N, x (both integers) 71

16152. (M. V. Arunachalan, M.A.)—Show that

$$[a(b-c)(bc-aa')(a^4-a'^4)]/(a-a') + [b(c-a)(ca-bb')(b^4-b'^4)]/(b-b') + [c(a-b)(ab-cc')(c^4-c'^4)]/(c-c') = (b-c)(c-a)(a-b)(bc-aa')(ca-bb')(ab-cc')\,S/abc$$

where $aa' = bb' = cc'$ and $S = a+b+c+a'+b'+c'$ 100

16155. (Professor Nanson.)—Find the locus of the foci of a conic of given eccentricity described about a given triangle 87

16157. (W. F. Beard, M.A.)—T is the pole of a chord PQ of a parabola whose focus is S; TS, PS, QS meet the circum-circle of the triangle TPQ in T', P', Q'. Prove that the tri ngles TPQ, T'Q'P' are congruent 92

16158. (V. Ramaswami Aiyar, M.A.)—ABC is a triangle inscribed in a conic, focus S. If the base BC passes through the reflection of the vertex A with respect to the S-directrix, then the pedal circle of the triangle ABC with respect to S has its diameter equal to SA 96

16160. (I. Arnold.)—Three points A, B, C are in one right line, and a circle is drawn with A as centre. Find the locus of the intersection of the right line passing through B and one extremity of a diameter of the circle and the right line passing through C and the other extremity of the same diameter 81

16161. (V. Daniel, B.Sc.)—The perpendiculars of A'B'C' are rotated in the same direction through an angle ϕ about the vertices, forming a triangle (modulus of similarity $1/\lambda \equiv 2\sin\phi$) capable of being revolved round some point O through an angle θ, so as to lie inscribed to Δ' (A on B'C', ...). Show that

$$4\sin\theta = \sqrt{(4\lambda^2-1)} \pm \sqrt{(4\lambda^{-2}-1)} \text{ ..... } 109$$

16163. (Professor Cochez.)—On donne trois cercles co-axiaux. Lieu de P tel que les tangentes PA, PB et PC à ces cercles (A, B, C étant les points de contact) vérifient la relation $(PA \cdot PB)/PC^2 = K$ 91

16165. (Professor R. W. Genese, M.A.)—A force acts at each vertex of a rigid polygon at right angles and proportional to the join of the adjacent vertices. Then (1) the system balances; (2) if each force retain its original direction, but the polygon be turned through an angle θ, the system is equivalent to a couple whose moment varies as $\sin\theta$ 86

16167. (W. J. Greenstreet, M.A.)—Solve

$$(y^3+2xy^2)y' + (x+y)(xy'-y) = 2y^3 \text{ ..... } 88$$

16168. (Lt.-Col. Allan Cunningham, R.E.)—If ${}^nC_x$ denote the number of combinations of n things taken x together, show that ${}^{p-1}C_x \equiv (-1)^x \pmod p$ when p is prime; and show how far this is true when p is composite. Also show that $\frac{1}{4}({}^{2q}C_q) - 1 \equiv 0 \pmod{q^3}$ when q is a prime > 3; when $q = 3$ the divisor is only q^2 94

16169. (R. W. D. Christie.)—When p is any prime of form $4m-1$ and r_1, r_2 are two different prime roots, prove that $r_1^2 + r_1 r_2^2 \equiv 0 \pmod p$. Give a few other similar equations and show their utility in factorization. *E.g.*, $10^2 + 10.3^2 \equiv 0 \pmod{19}$; also

$$2^2+2.13^2 = 3^2+3.15^2 = 13^2+13.14^2 \ldots \text{ ..... } 102$$

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16171. (M. V. Arunachalan, M.A.)—Find the value of the infinite products (1) $(1+x^4/1^4)(1+x^4/2^4)(1+x^4/3^4)$, ...,
(2) $(1+x^4/1^4)(1+x^4/3^4)(1+x^4/5^4)$, ... 95

16176. (Professor Lauvernay.)—Construire la courbe
$\rho^2-4\rho\cos\frac{1}{2}\omega+1=0$ 103

16183. (Professor R. W. Genese, M.A.)—Inward forces act at the mid-points of the sides of any plane quadrilateral at right angles and proportional to the opposite sides. Prove (1) that they balance, (2) that the resultant of two opposite forces acts along the directrix of the parabola inscribed in the quadrilateral 97

16200. (Professor Sanjána, M.A.)—The great circle bisecting perpendicularly the side BC of the spherical triangle ABC meets the great circles bisecting internally and externally the angle A in the points D, E. Prove that D and E lie on the small circle circumscribing ABC 105

ERRATA.

P. 23, l. 12 from foot, *for* "one" *read* "(1)."

P. 24, l. 17, *for* "$_{m+n}C^m$" *read* "$_{m+n}C_m$."

P. 41, l. 1 should *read* "Étant donné un paraboloïde."

P. 46, l. 3, *read* "E, F; E′, F′; E″, F‴."

P. 48, l. 7 from foot, *for* "P" *read* "P′."

P. 51, l. 14, *for* "roves" *read* "proves."

P. 52, l. 3 from foot, *for* "A.N." *read* "A.M."

P. 61, Result 8, *read* "t^4 [*not* t^2] $= u_1^2+u_4^2$."
Insert "N_1" to left of present result (9*a*).
Change the numbers of the two results "(9*a*), (10)" to "(10), (11)."
In the examples *read* "$2\cdot7^2$" *instead of* "27^2."
In the table—With $p=233$, *read* "$u_1=14$"; with $p=313$ *cancel* "$t_1=5$, $u_1=16$, $t_2=25$, $u_2=4$"; with $p=353$, *cancel* "$t_2=19$, $u_2=2$."

P. 71, l. 1 of Solution to Quest. 16150, *for* "N" *read* "N^2."

P. 74, l. 3 of Solution to Quest. 16144, *for* "tangent at P" *read* "tangent at *p*."

P. 86, l. 2, *for* second bracket *read* "$(b-c)$"; and l. 9, *insert* "$-$" before "c^2."

P. 100, Quest. 16152, in right-hand member *insert* "$-$" at commencement, and similarly in final result and Note.

P. 108, ll. 1 and 6, *for* "Proposer" *read* "writer."

P. 111, l. 17 from foot, *for* "$_7$" *read* "$\frac{3}{7}$."

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MATHEMATICS

FROM

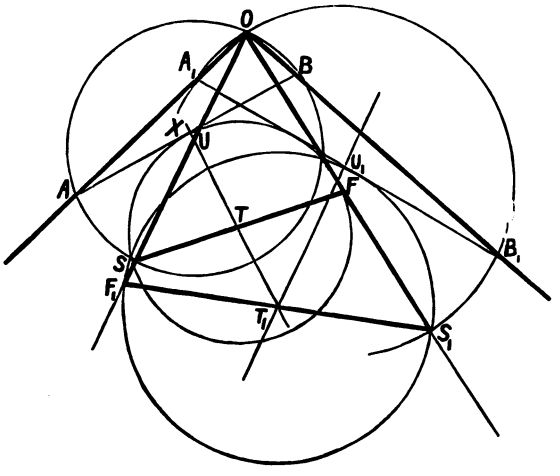
THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

16068. (C. E. YOUNGMAN, M.A.)—A parabola passing through a fixed point F touches two fixed lines OA, OB, and its focus S is inverted into P by the circle with centre O and radius OF. Prove that P lies on a parabola with focus F, touching OA and OB.

Solution by HENRY RIDDELL, M.E.

A particular case of a more general theorem. If a parabola with focus S touches OA and OB and passes through F, then, if S and F be inverted with any radius from O into F_1 and S_1 respectively, a parabola with focus



S_1 touching OA and OB passes through F_1. On OS as diameter describe

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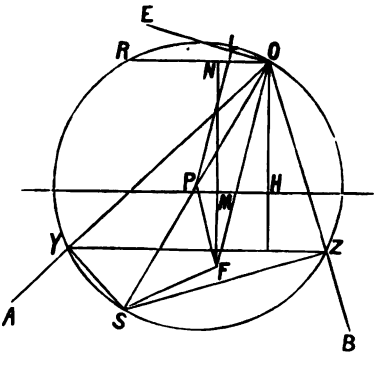
a circle cutting the tangents in A and B; then is AB the tangent at the vertex, and, by a well known property of the parabola, the circle on SF as diameter touches AB. Also, if S be a focus of a parabola and the circle on SF as diameter touches AB, then is F a point on the parabola. Inverting into F_1 and S_1, and on OS_1 as diameter describing a circle, A_1B_1 is the tangent at the vertex of the parabola touching OA and OB and having S_1 for focus: the parabola also passes through F_1. $F_1S_1/OS_1 = FS/OS$; therefore also $AB/A_1B_1 = SF/S_1F_1$. Since these ratios hold, A_1B_1 will touch the circle on S_1F_1 as diameter, if the angle between F_1S_1 and T_1U_1, the line of centres of the related circles, is equal to the angle between FS and TU. But the angle between the axis of the first parabola and OF or TU is the difference between the angles AOS and BOF. That between the axis of the second parabola and OF_1 or T_1U_1 is again the difference between AOS and BOF, and, since the tangents at the vertex are perpendicular to the axes, the angle between TU and AB is equal to that between T_1U_1 and A_1B_1, and the circle on S_1F_1 touches A_1B_1.

The following Solution of the theorem is due to Mr. R. F. Davis, M.A.:—

Draw SY, SZ perpendiculars on OA, OB respectively; then YZ is the tangent at the vertex of the first parabola, and a line through H (the orthocentre of the triangle OYZ) parallel to YZ is the directrix. A line OR through O parallel to YZ will be parallel to the directrix; and $\angle AOR = \angle$ (?)Z $\angle OSZ$ = complement of $\angle BOS$. Similarly, a line OE making $\angle AOE$ = complement of $\angle BOF$ will be parallel to the directrix of the second parabola, focus F.



Now $$\angle POL = (\angle AOB - \angle BOS) + (90° - \angle BOF)$$
$$= (\angle AOB - \angle BOF) + (90° - \angle BOS)$$
$$= \angle FOY + \angle AOR = \angle FON.$$

Since $OP.OS = OF^2$,

$$FP : FS = OP : OF [= PL : FN] = OF : OS.$$

Hence $$PL - FP : FN - FS = OF : OS.$$

But $FN - ES = FN - FM = OH$; therefore $PL - FP = OF(OH/OS)$ = constant, since the vertical angle YOZ is constant.

Thus $FP = PL - \text{constant}$; and the locus of P is the second parabola.

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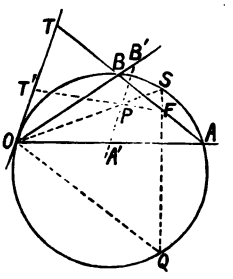
[*Note.*—The PROPOSER thinks it easier to prove his theorem as it stands, and then extend it into Mr. Riddell's by proportional radii from O. Thus: let AFB be the tangent at F to the first parabola, and OQ, AB parallel chords in the circle OABS. Then OB and AF subtend equal angles at S; therefore likewise AQ and AF; and S, F, Q are collinear. Also, if OT, touching the circle at O, cuts AB at T, the points S, F, O, T are concyclic. Upon inversion, OABS becomes a line A′B′P parallel to OT; SFOT becomes a line PFT′, and AFBT a circle OA′FB′T′. Hence a parabola with focus F can be inscribed in the triangle OA′B′, touching A′B′ where FT′ cuts it: that is, at P.]



16062. (ROBT. W. D. CHRISTIE.)—Why is the remainder of

$$100000010000001/109$$

or
$$100000010000001 \pmod{109} = 1 + 10 \cdot 19^2 + (10 \cdot 19^2)^2$$
$$= 1 + 13 + 13^2 \pmod{109}$$
$$= 74 \pmod{109}\ ?$$

Generalize for any prime divisor.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

The given large number $N = 10^{14} + 10^7 + 1$. Now $10^2 \equiv 19$, $10^6 \equiv 19^2$, $10^7 \equiv 10 \cdot 19^2 \equiv 13 \pmod{109}$; therefore

$$10^{14} \equiv (10 \cdot 19^2)^2 \equiv 13^2 \equiv 60 \pmod{109};$$

therefore $N \equiv (10 \cdot 19^2)^2 + 10 \cdot 19^2 + 1 \equiv 13^2 + 13 + 1 \equiv 74$.

The generalization appears to be: Let $N = a^{2n} + a^n + 1$ and let $a^n \equiv a \pmod{p}$. Then $N \equiv a^2 + a + 1 \pmod{p}$.

16040. (W. J. DOBBS, M.A.)—A bullet was projected from a point on a horizontal plane, and just grazed the top of a wall 1,000 yards distant. It struck the plane 10 yards beyond the wall, which is 10 feet high. Neglecting the air resistance, calculate the velocity of projection, the time of flight, and the greatest height above the plane to which the bullet ascended.

Solution by A. H. GILLSON.

Let u be the velocity of projection in feet per second, and θ the angle of projection. If T be the time that elapses before the bullet passes near the wall, then

$$3000 = u \cos\theta \cdot T, \quad 10 = u \sin\theta \cdot T - \tfrac{1}{2}g \cdot T^2 \ldots\ldots\ldots (1, 2).$$

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Eliminate T from (1) and (2), and

$$10w^2 = u^2 \tan\theta \cdot 3000 - \tfrac{1}{2}g \cdot 3000^2 \sec^2\theta \quad \ldots\ldots\ldots\ldots (3).$$

Also range on horisontal plane $= (2u^2 \sin\theta \cos\theta)/g = 3030$ ..... (4).

Substitute this value of u^2 obtained from (4) in (3), and multiply through by $2\cos^2\theta$, and $\theta = \tan^{-1} \frac{101}{150}$ ..... (i.).

Substitute the value of $\sin\theta$ and $\cos\theta$ obtained from (i.) in (4); then (taking with the Proposer the value of g as 32·2 ft./sec^2, and working the results to three significant digits) we get $u^2 = \frac{202}{3}\frac{800}{3}$ or $5010g$; therefore $u = 402$ feet per second approx. ..... (ii.).

Also greatest height attained $= (u^2 \sin^2\theta)/2g = 255$ feet .....(iii.).
Also, if t be the time of flight,

$t = (2u \sin\theta)/g = 7{\cdot}96$ seconds .....(iv.).

16070. ("Cymro.")—Prove geometrically that a chord of an ellipse which subtends a constant angle at the focus always touches an ellipse with the same focus and directrix.

Additional Solution by R. F. Davis, M.A.

(i.) If the tangent at a point P of a conic (focus S) intersects the directrix KK′ in Z and PM is perpendicular to KK′, then ZP divides the angle SZK in such a manner that the perpendiculars from any point on ZP on ZS, ZK are in a constant ratio (e : 1). Conversely, if S is a given point, KK′ a given line, Z any variable point on KK′, then the envelope of a line through Z dividing the angle SZK in such a manner that the sine of one part is in a constant ratio to the sine of the other part is a conic having S for focus, KK′ for directrix, and the constant ratio for eccentricity.

(ii.) Now let PQ be a chord of a conic subtending a constant angle at the focus S. Produce PQ to meet the directrix KK′ in R, and draw PM, PL perpendiculars on KK′, SR respectively. Then SP : PM is constant. Also PL : SP is constant; for PSL is the complement of half the given angle. Hence PL : PM is constant, and the envelope of PQ is a conic, focus S, directrix KK′.

15949. (C. Bickerdike.)—In Price's *Infinitesimal Calculus*, Vol. iv., p. 129, the following question is proposed:—Show that $w^2a/6E$ is the work done when a heavy elastic string of length a and weight w hanging vertically is stretched by its own weight. Is the statement correct?

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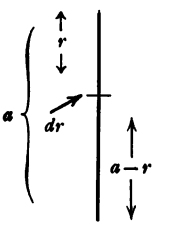
Solution by ERNEST W. HILL.

Consider any element of the string dr at a distance r from the point of support. The lengthening of this element dr due to the stress caused by the mass of the remainder of the string of length $a-r$ will be $dr\,(w/E)[(a-r)/a]$, and the work done by reason of the point of application of the force $w[(a-r)/a]$ being moved through a distance $dr\,(w/E)[(a-r)/a]$ will be $(w^2/E)[(a-r)/a]^2\,dr$; therefore



$$\text{total work done} = \int_{r=0}^{r=a} \frac{w^2}{E}\left(\frac{a-r}{a}\right)^2 dr = \frac{w^2}{Ea^2}\int_0^a (a^2-2ar+r^2)\,dr$$

$$-\left[a^2r-\tfrac{1}{2}(2ar^2+\tfrac{1}{3}r^3)\right]_0^a = \frac{w^2a}{3E}.$$

Thus the expression $w^2a/6E$ is not correct.

16094. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Let

$$E_n = 2^{2^n},\quad F_n = E_n+1,\quad G_n = E_n-1.$$

Show that every F_n (>5) may be expressed algebraically in the form

$$2F_n = t^2-G_n\,.\,u^2,$$

and obtain the expression. Discuss the possibility of obtaining a second expression $2F_n = t'^2-G_n\,.\,u'^2$ which, together with the former, shall be suitable for factorizing F_n.

Solution by the PROPOSER.

Here $E_n = E_{n-1}^2$, and $G_n = F_0\,.\,F_1\,.\,F_2 \ldots F_n$, and note that $n>1$, since $F_n>5$. Hence every F_n (>5) is of form $(8m+1)$ and every factor of every F_n is of same form; hence both ± 2 are 2-ic residues of every F_n (>5). Now $G_n \equiv -2 \pmod{F_n}$, and $2F_n \equiv +4 \pmod{G_n}$. Hence G_n is a 2-ic residue of F_n, and $2F_n$ is a 2-ic residue of G_n. Hence $2F_n$ is certainly expressible in the principal genus of the determinant $+G_n$, and possibly in the principal class ($t = G_n\,.\,u^2$).

To investigate this, try F_2, F_3, F_4; these give
$2\,.\,17 = 7^2-15\,.\,1^2$, $2\,.\,257 = 53^2-255\,.\,3^2$, $2\,.\,65537 = 3857^2-65535\,.\,15^2$.
These give the clue to the general (algebraic) forms

$$2F_n = (G_{n-2}\,.\,F_{n-1}+2)^2-G_n\,.\,G_{n-2}^2 = t^2-G_n\,.\,u^2,$$

which will be found to reduce to $2F_n$, as it should. Moreover, this is the base form (*i.e.*, t, u are minima). For the unit form of the determinant $+G_n$ is $\tau^2-G_n\,.\,v^2 = +1$, where $\tau = E_{n-1}$, $v=1$, and the above values of t, u satisfy the condition of minima, viz., $t/u > (\tau+1)/v$ [see result (13), Art. 14 of the Introduction to the writer's Tables of *Quadratic Partitions*, 1904].

Next, as to the chance of factorizing F_n by the discovery of a second, but independent, partition $(t'^2-G_n u'^2)$. The determinant $(+G_n)$ is not really suitable for this purpose; for it may be shown* that $+G_n$ is

* For proof of this the writer is indebted to the Rev. J. Cullen.

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a determinant containing more than one class per genus (when $G_n > 255$); so that there is no *a priori* certainty that a second (independent) partition exists (even when F_n is really composite), and the non-existence of a second (independent) partition (if it could be proved) would not suffice to prove F_n prime. Of course a second (independent) partition—if it could be found—would suffice to factorise F_n. But the difficulty of finding this second partition would certainly be very great, as the numbers F_n now requiring examination are so large. And note that the minimum t', u' would have to be greater than the t, u above found.

15887 & 15944. (Professor F. ROGEL.)—(15887) To find directly the common tangents of two conic sections having a common focus.

(15944) To find directly the cuts of two conic sections having a common focus.

Solution by C. E. YOUNGMAN, M.A.

Reciprocate the conics from their common focus S, getting two circles (the inverses of the two auxiliary circles, which we may suppose given); note the common points and tangents of these, and reciprocate back again. This seems the obvious method, and the drawing of it would be simple enough, especially if the common tangents of the circles are obtained by ruler and eye—probably the most accurate way, though not strictly geometrical. In Question 15887 it would only be necessary to take Y, Y′ where the auxiliary circles meet, and there draw perpendiculars to SY, SY′.

Question 15944 may be regarded also as a problem in "tangencies": for the required "cuts" are the centres of circles passing through S and touching two given circles: those, namely, whose centres are F, F′, the other foci, and radii the corresponding transverse axes, of the conics. A convenient construction is the following, slightly modified from Vieta's, so as not to need the centres of similitude:—Draw FF′, giving AB and A′B′ diameters of the two circles (let AB and A′B′ have the same direction); find a, b, a', b', the centres of the circles SAA′, SBB′, SA′B, SAB′; then R, R′, the radical centres of the systems $(a)(b)(F)$ and $(a')(b')(F)$; and draw the two orthogonal circles, cutting (F) at p, p' and q, q'. Then, where Fp, Fp' meet ab, and Fq, Fq' meet $a'b'$, lie the required points.

The lines ab, $a'b'$ meet, it is known, at O, the radical centre of the point-circle S and the circles F, F′; but the radical axis of S and (F) is the polar of S with respect to one of the auxiliary circles; it is therefore the S-directrix of one of the conics; hence O is the point common to the two S-directrices. We find then that ab, $a'b'$ must be the locus of points whose distances from these directrices are inversely as the eccentricities of the conics; and they might be constructed from that property.

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16009. (Professor COCHEZ.)—Résoudre

$$(ax)^{\log a} = (by)^{\log b}, \quad b^{\log x} = a^{\log y}.$$

Solution by R. J. WHITAKER, J. G. M. B. A., *and others.*

Taking logarithms,

$$\log a\,(\log a + \log x) = \log b\,(\log b + \log y) \quad \text{.....(i.)},$$

$$\log x \log b = \log y \log a \quad \text{.....(ii.)},$$

or $\log a \log x - \log b \log y = (\log b)^2 - (\log a)^2$

and $\log b \log x - \log a \log y = 0.$

Multiply by $\log b$, $\log a$, and subtract; then

$$\log y\,[(\log a)^2 - (\log b)^2] = \log b\,[(\log b)^2 - (\log a)^2];$$

therefore $\log y = -\log b$, unless $a = b$; *i.e.*, $y = 1/b$; similarly $x = 1/a$.

Note.—Some of the Solvers remark that when $a = b$ then $x = y$ = any positive quantity whatever.

16014. (R. F. DAVIS, M.A.)—Where is there to be found an investigation of the locus of the focus of a parabola which passes through three given points?

Discussion by Professor NANSON.

The problem has been discussed by Salmon, Burnside, and Hirst (see *Reprint*, First Series, Vol. IV., p. 19, and Salmon's *Conics*, 5th Ed., pp. 389, 390.

Taking the three fixed points as reference points, any conic through them is $f/x + g/y + h/z = 0$, and the condition that it touches the line l, m, n is $\surd(fl) + \surd(gm) + \surd(hn) = 0$. It at once follows that the locus of the meet P of tangents from two fixed points P′, P″ to a conic through the three fixed points and touching the line l, m, n is

$$\begin{vmatrix} l, & m, & n \\ \surd X', & \surd Y', & \surd Z' \\ \surd X'', & \surd Y'', & \surd Z'' \end{vmatrix} = 0 \quad \text{.....(1)},$$

where X, Y, Z are the co-factors of x, y, z in the determinant (x, y, z). When rationalized one is of order 16 in x, y, z. But the locus is in reality of order 12 (Salmon, *l.c.*). The apparent discrepancy is accounted for by the fact that the rationalized equation is satisfied when $(x, y, z) = 0$, because then $X' : Y' : Z' = X'' : Y'' : Z''$. The inference is that the rationalized equation must be divisible by $(xyz)^4$.

Now, making the line l, m, n coincide with P′P″, we get for the locus of the meet P of tangents from P′, P″ to conics touching P′P″, and passing through the three reference points the curve

$$(\surd X, \surd Y, \surd Z) = 0$$

of order 16. But the line P′P″ is now manifestly part of the locus. The remaining part is (Salmon, *l.c.*) of order 8, and the inference is that the rationalized equation is now divisible by $(x, y, z)^8$. Taking

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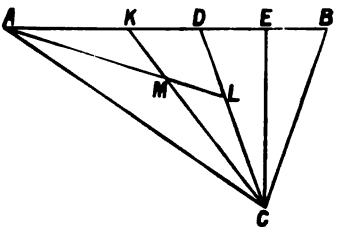
P', P'' at the focoids, we get the locus of the focus of a parabola through three fixed points. It was an analogous treatment of the locus of the focus of a parabola touching three given lines which suggested Question 15146, *Reprint*, New Series, Vol. III., p. 108.

15929. (Professor SANJÁNA, M.A.)—The straight line AB is divided at D so that $AB \cdot BD = AD^2$; with centres A and D and radii AB and DA respectively, circles are described, meeting in C. Prove by the first two books of Euclid that BC = AD, and that straight lines joining C to the in-centre and circum-centre of ABC cut AB isotomically.

[The first part solves Euclid IV. 10 without using the third and fourth books.]

Solution by HENRY RIDDELL, M.E., *and others.*



AB is divided at D, so that

$$AB \cdot BD = AD^2.$$

Bisect DB at E and erect a perpendicular. Make DC = DA; then is BC also = DA. Then

$$\begin{aligned} AC^2 &= AE^2 + EC^2 \\ &= AD^2 + DE^2 + EC^2 + 2AD \cdot DE \\ &= 2AD^2 + AD \cdot DB \\ &= AD^2 + AD \cdot AB \\ &= AB \cdot BD + AB \cdot AD = AB^2. \end{aligned}$$

Hence the circle with centre A and radius AB will pass through C. Then $\angle DBC = \angle CDB = 2\angle CAB$, thus solving Euclid IV. 10. Now CD passes through the centre of the inscribed circle, since it evidently bisects the angle at C. Bisect the angle at A, cutting CD at L; then L is the centre of the inscribed circle, and, since AB = AC, AL also passes through M, the centre of the described circle. Join CM and produce to K. Then $\angle LAC = \angle KCA$, and (by Euclid I. 26) CL = AK. But, since $\angle CBD = \angle CDB = 2\angle DCB$, BL, bisecting $\angle DBC$, = LC = BD, and BD = AK.

15951. (A. M. NESBITT, M.A.)—If p be the probability that an event happens m times out of $m+n$ trials, find the chance that it will happen μ times out of $\mu+\nu$ trials.

Correction by D. M. Y. SOMMERVILLE, M.A., D.Sc.

The solution of this problem (as given in the *Reprint*, New Series, Vol. X., p. 110) requires correction. That there is an error in it will be readily seen. The Solver, in fact, assumes that the probability of an

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event happening exactly m times out of $m+n$ is the same as the probability of its happening exactly Km times out of $K(m+n)$. This is not true. For example, the chance that of two throws with a coin one will be a head is $\frac{1}{2}$; the chance that two out of four will be heads is only $\frac{3}{8}$. The fallacy of this assumption is very obvious when we make $m = n$. The formula for the probability of the event happening in any case becomes then

$$q = [pm + (1-p)n]/(m+n) = n/(m+n) = \tfrac{1}{2};$$

i.e., whatever is the probability of an event happening exactly m times out of $2m$, the probability of its happening in any case is $\frac{1}{2}$!

The problem may be investigated as follows:—Let q be the probability of the event happening in any case, and consider $m+n$ trials. The chance that the first is a success is q, and the chance that it is a failure is $1-q$. The chance that the first m are successes and the last n failures is $q^m(1-q)^n$. Hence, since the successes may occur in ${}_{m+n}C_m$ different orders, the total chance of m successes and n failures is

$$p = {}_{m+n}C^m\, q^m (1-q)^n.$$

Similarly,
$$s = {}_{\mu+\nu}C_\mu\, q^\mu (1-q)^\nu.$$

Eliminating q between these two equations, we get s in terms of p, m, n, μ, ν.

16082. (R. CHARTRES.)—Sum the series

$$1 - \frac{C_1}{3} + \frac{C}{6} - \frac{C_3}{10} + \frac{C_4}{15} - \ldots$$

where $C_r \equiv$ combinations of n things r together.

Solutions (I.) *by* R. F. DAVIS, M.A.; (II.) *by* M.V. ARUNACHALAN, M.A.

(I.) Denote the series to r terms by Σ; and in the first case suppose r is odd. Then

$$\tfrac{1}{2}\Sigma = \frac{1}{1.2} - \frac{1}{2.3}\binom{n}{1} + \frac{1}{3.4}\binom{n}{2} - \frac{1}{4.5}\binom{n}{3} + \ldots \text{ to } r \text{ terms.}$$

Therefore

$$\tfrac{1}{2}(n+1)(n+2)\,\Sigma = \binom{n+2}{2} - \binom{n+2}{3} + \binom{n+2}{4} - \ldots \text{ to } r \text{ terms.}$$

But
$$(1+x)^{n+2} = 1 + \binom{n+2}{1}x + \binom{n+2}{2}x^2 + \ldots,$$
$$(1+x)^{-1} = 1 - x + x^2 - \ldots.$$

Multiply these series together and equate the coefficients of x^{r+1} on both sides. Thus

$$\binom{n+1}{r+1} = 1 - \binom{n+2}{1} + \binom{n+2}{2} - \ldots + (r+2) \text{ terms}$$
$$= -(n+1) + \tfrac{1}{2}(n+1)(n+2)\,\Sigma$$

or
$$\Sigma = \frac{2}{(n+1)(n+2)}\left\{n+1+\binom{n+1}{r+1}\right\}.$$

Next let r be even. The result is the same, with the exception that the inner bracket is negative. In either case when the series runs its full course the sum $= 2/(n+2)$.

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(II.) Let $f(n) \equiv 1 - \frac{C_1}{3} + \frac{C_2}{6} - \frac{C_3}{10} + \frac{C_4}{15} - \ldots$

$$= 1 - \frac{n}{3} + \frac{n(n-1)}{1.2}\,\frac{1}{6} - \frac{n(n-1)(n-2)}{1.2.3}\,\frac{1}{10} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\,\frac{1}{15} - \ldots \quad \ldots\ldots\ldots\ldots(1),$$

$$f(n) = 1 - \frac{n}{3} + \frac{n(n-1)}{1.2}\,\frac{1}{6} - \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Subtracting (2) from (1), we have

$$0 = 1 - \frac{n+3}{3} + \frac{n(n+3)}{1,2.6} - \frac{n(n-1)(n+3)}{1.2.3.10} + \ldots$$

$$= 1 + \frac{n+3}{n+1}\left\{ -\frac{n+1}{3} + \frac{(n+1)n}{1.2}\,\frac{1}{6} - \frac{(n+1)n(n-1)}{1.2.3}\,\frac{1}{10} + \ldots \right\}$$

$$= 1 + \frac{n+3}{n+1}\{f(n+1) - 1\};$$

therefore $f(n+1) = \frac{2}{n+3}$; therefore $f(n) = \frac{2}{n+2}$.

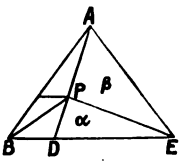
The following Solution is due to the Proposer :—

$$AP/AD = x, \quad \Delta = 1.$$

$$M(x^n) = M(1-\alpha)^n = 1 - \frac{c_1}{3} + \frac{c_2}{6} - \frac{c_3}{10} + \ldots$$

(by Whitworth's theorem), since

$$1 = \Delta = \alpha + \beta + \gamma.$$



But $M(x^n) = \int_0^1 x^n x\,dx \Big/ \int_0^1 x\,dx = \frac{2}{n+2}$;

therefore $1 - \frac{c_1}{3} + \frac{c_2}{6} - \frac{c_3}{10} + \ldots = \frac{2}{n+2}$.

15498. (G. H. Hardy, M.A.)—Prove that the reciprocal of the product of two determinants is the product of their reciprocals.

Solution by C. M. Ross.

Take the determinant

$$\Delta \equiv \begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \ldots & \ldots & & \ldots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Its reciprocal is

$$\Delta' \equiv \begin{vmatrix} X_{11} & X_{12} & \ldots & X_{1n} \\ X_{21} & X_{22} & \ldots & X_{2n} \\ \ldots & \ldots & & \ldots \\ X_{n1} & X_{n2} & \ldots & X_{nn} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

X_{rs} in (2) being the co-factor of x_{rs} in (1).

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Multiplying together (1) and (2), the result is

$$\Delta\Delta' = \begin{vmatrix} x_{11}X_{11}+\ldots+x_{1n}X_{1n} & x_{11}X_{21}+\ldots+x_{1n}X_{2n} & \ldots & x_{11}X_{n1}+\ldots+x_{1n}X_{nn} \\ x_{21}X_{11}+\ldots+x_{2n}X_{1n} & x_{21}X_{21}+\ldots+x_{2n}X_{2n} & \ldots & x_{21}X_{n1}+\ldots+x_{2n}X_{nn} \\ \ldots\quad\ldots\quad\ldots & \ldots\quad\ldots\quad\ldots & \ldots & \ldots\quad\ldots\quad\ldots \\ x_{n1}X_{11}+\ldots+x_{nn}X_{1n} & x_{n1}X_{21}+\ldots+x_{nn}X_{2n} & \ldots & x_{n1}X_{n1}+\ldots+x_{nn}X_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} \Delta & 0 & \ldots & 0 \\ 0 & \Delta & \ldots & 0 \\ \ldots & \ldots & \ldots & \\ 0 & 0 & \ldots & \Delta \end{vmatrix} = \Delta^n;$$

therefore $\Delta' = \Delta^{n-1}$; *i.e.*, the reciprocal of any determinant of the n-th order is equal to the $(n-1)$-th power of the latter. Hence, if Δ denote the product of two n-th order determinants Δ_1 and Δ_2, and D the reciprocal of their product, then

$$D = \Delta^{n-1} = \{\Delta_1\Delta_2\}^{n-1} = \Delta_1^{n-1}\Delta_2^{n-1}$$

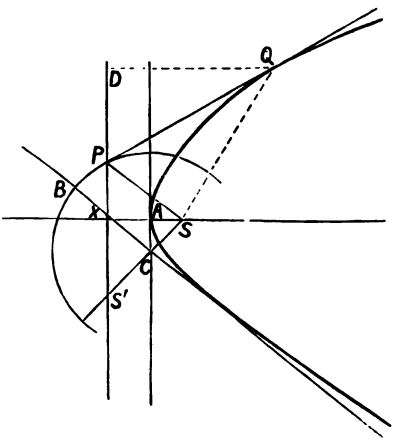
= product of the reciprocals of Δ_1 and Δ_2.

16032. (H. L. TRACHTENBERG, B.A.)—A parabola has its focus at a focus of an ellipse and touches its minor axis. Show that the points of contact of a common tangent subtend a right angle at the common focus.

Remark by Professor SANJÁNA, M.A.

The following is an elementary solution. Through the focus S of a parabola draw a straight line to meet the tangent at the vertex and the directrix in C and S′; then SC = CS′. With S and S′ as foci draw any ellipse; then CB, the minor axis, perpendicular to SC, of course touches the parabola. Let S′X meet the ellipse in P, and draw PQ touching the parabola. By the property of the parabola,

$\angle SPQ = \angle DPQ$,



so that PQ is equally inclined to PS and PS′; it is therefore a tangent to the ellipse. And it is known that the angle PSQ is a right angle.

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16078. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Express $F_5 = 2^{32}+1$ and $F_6 = 2^{64}+1$ each in two different ways in each of the 2-ic forms

$$(a^2+b^2),\quad (c^2+2d^2),\quad (e^2-2f^2).$$

Solution by T. STUART, M.A., D.Sc.

We have in general

$$a^4+b^4 \equiv (a^2)^2+(b^2)^2 \equiv (a^2-b^2)^2+2(ab)^2 \equiv (a^2+b^2)^2-2(ab)^2,$$

and hence in particular

$$F_5 = 2^{32}+1 \equiv (2^8)^4+1^4 \equiv (2^{16})^2+1^2 \equiv (2^{16}-1)^2+2(2^8)^2$$
$$\equiv (2^{16}+1)^2-2(2^8)^2,$$

with similar representations for F_6.

Now 641 (the lowest factor in F_5)

$$= 25^2+4^2 = 21^2+2\cdot 10^2 = 29^2-2\cdot 10^2;$$

therefore

$$\frac{F_5}{641} = \frac{(2^{16})^2+1^2}{25^2+4^2} = \left(\frac{25\cdot 2^{16}-4}{641}\right)^2+\left(\frac{4\cdot 2^{16}+25}{641}\right)^2 = 2556^2+409^2.$$

Hence
$$F_5 = (2556^2+409^2)(25^2+4^2)$$
$$= (2556\cdot 25\pm 409\cdot 4)^2+(409\cdot 25\mp 4\cdot 2556)^2;$$

i.e.,
$$F_5 = 65536^2+1^2,\ 62264^2+20449^2.$$

Also
$$\frac{F_5}{641} \equiv \frac{(2^{16}-1)^2+2(2^8)^2}{21^2+2\cdot 10^2}$$
$$= \left(\frac{21(2^{16}-1)+10\cdot 2^8}{641}\right)^2+2\left(\frac{10(2^{16}-1)-21\cdot 2^8}{641}\right)^2$$
$$= 2155^2+2\cdot 1014^2.$$

Hence
$$F_5 = (2155^2+2\cdot 1014^2)(21^2+2\cdot 10^2)$$
$$= 24975^2+2\cdot 42844^2 = 65535^2+2\cdot 256^2;$$

and, lastly, we have

$$\frac{F_5}{641} = \frac{65537^2-2\cdot 256^2}{29^2-2\cdot 10^2}$$
$$= \left(\frac{29\cdot 65537+20\cdot 256}{641}\right)^2-2\left(\frac{10\cdot 65537+29\cdot 256}{641}\right)^2$$
$$= 2973^2-2\cdot 1034^2.$$

Hence
$$F_5 = (2973^2-2\cdot 1034^2)(29^2-2\cdot 10^2)$$
$$= 65537^2-2\cdot 256^2,\ 106897^2-2\cdot 59716^2.$$

These are the only representations of the required forms, since F_5 has only two factors.

In a similar manner, since

$$274177 \text{ (the lowest factor in } F_6) = 516^2+89^2 = 523^2+2\cdot 18^2$$
$$= 575^2-2\cdot 168^2,$$

we find that

$$F_6 = (2^{32})^2+1^2 = 4046803256^2+1438793759^2$$
$$= (2^{32}-1)^2+2(2^{16})^2,\ 4274656527^2+2\cdot 295004788^2$$
$$= (2^{32}+1)^2-2(2^{16})^2,\ 6063569153^2-2\cdot 3026559664^2,$$

which are the representations asked for in the Question.

Note.—In order to obtain the second set of partitions of a given number N when one set are already known as well as those of the

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smallest factor, it is quite unnecessary to find the corresponding partitions of the larger factor. For, if $N = n_1 n_2 = P_0^2 + kQ_0^2$ (where P_0, Q_0, and k are known), and if $n_1 = p^2 + kq^2$ ($n_1 < n_2$), then

$$Nn_1^2 \equiv (P_0^2 + kQ_0^2)\,[(p^2 - kq^2)^2 + k(2pq)^2],$$

and hence

$$N = \left(\frac{P_0(p^2 - kq^2) \pm 2Q_0 kpq}{n_1}\right)^2 + k\left(\frac{Q_0(p^2 - kq^2) \mp 2P_0 pq}{n_1}\right)^2,$$

which is the second partition required. By this means we are enabled to shorten the process greatly. Thus, for example, in one of the cases already examined (F_5), we have $641 = 21^2 + 2 \cdot 10^2$; therefore

$$(641)^2 = (21^2 - 2 \cdot 10^2)^2 + 2(20 \cdot 21)^2 = (241)^2 + 2(420)^2.$$

Hence
$$F_5 = \left(\frac{(2^{16} - 1)\,241 + 420 \cdot 2^9}{641}\right)^2 + 2\left(\frac{(2^{16} - 1)\,420 - 241 \cdot 2^8}{641}\right)$$
$$= 24975^2 + 2 \cdot 42844^2,$$

as already obtained.

When $k = Q_0 = 1$ it is easily seen from the fact that $(p^2 - kq^2)/n_1$ and $2pq/n_1$ are *proper* fractions that the second partition of N in the form $P^2 + Q^2$ is given by

$$P = 1 + I\left(\frac{2P_0 pq}{n_1}\right), \qquad Q = I\left(\frac{P_0(p^2 - q^2)}{n_1}\right),$$

where I(A) means the integral part of A.

For example, in the case of F_5, we have $p = 25$, $q = 4$, $P_0 = 2^{16}$; and therefore $F_5 = P^2 + Q^2$, where

$$P = 1 + I\left(\frac{5^2}{641} 2^{19}\right) = 20449, \qquad Q = I\left(\frac{621}{641} 2^{16}\right) = 62264.$$

Similarly, for $F_{4096} = 2^{2^{12}} + 1$, of which the lowest factor* is

$$114689 = 260^2 + 217^2,$$

we see that the values of P and Q are given by

$$P = 1 + I\left(\frac{5 \cdot 7 \cdot 13 \cdot 31}{114689} 2^{2051}\right), \qquad Q = I\left(\frac{3^2 \cdot 43 \cdot 53}{114689} 2^{2048}\right).$$

The PROPOSER's more general discussion of the Question is as follows:—

Let $E_n = 2^{2^n}$, $F_n = E_n + 1$, $G_n = E_n - 1$; then $E_n = E_{n-1}^2$.

All numbers of form $N = (x^4 + y^4)$ are susceptible of three 2-ic partitions algebraically, viz.,

$$N = a^2 + b^2 = c^2 + 2d^2 = e^2 - 2f^2$$
$$= (x^2)^2 + (y^2)^2 = (x^2 - y^2)^2 + 2(xy)^2 = (x^2 + y^2)^2 - 2(xy)^2.$$

These may be called (for shortness) the (a, b), (c, d), (e, f) partitions. Also, if N be composite, it is always susceptible of a second set of such partitions, arithmetically (*i.e.*, not algebraically): these may be called for shortness the (a′, b′), (c′, d′), (e′, f′) partitions.

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* The discovery of the factor is due to M. Pervouchine, *Nature*, Vol. XVIII., p. 104.

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Now, let $N = q \cdot Q$, and let the partitions of q, Q similar to the above be denoted by

$$q = \alpha^2 + \beta^2 = \gamma^2 + 2\delta^2 = \eta^2 - 2\phi^2,$$
$$Q = a^2 + b^2 = c^2 + 2d^2 = e^2 - 2f^2,$$

and let q be so small that its three 2-ic partitions (of above kind) can be easily found.

Then the corresponding 2-ic partitions of Q can always be found—even when Q is very large—by the process of "conformal division" (for full explanation of this process, see the Proposer's paper on the "Connexion of 2-ic Forms," Art. 15 *et seq.*, in *Proc. Lond. Math. Soc.*, Vol. XXVIII., 1896) from the data

$$a^2 + b^2 = \frac{\alpha^2 + b^2}{\alpha^2 + \beta^2}, \quad c^2 + 2d^2 = \frac{c^2 + 2d^2}{\gamma^2 + 2\delta^2}, \quad c^2 - 2f^2 = \frac{c^2 - 2f^2}{\eta^2 - 2\phi^2}.$$

The three 2-ic partitions of the factors q, Q being now known, the corresponding partitions may be compounded by the process of conformal multiplication (for full explanation of this process, see the Proposer's paper, above quoted, Art. 6), thus

$$qQ = (\alpha^2 + \beta^2)(a^2 + b^2) = (\gamma^2 + 2\delta^2)(c^2 + 2d^2) = (\eta^2 - 2\phi^2)(e^2 - 2f^2).$$

The process yields two product partitions in each case: one of these will be simply the known set N = (a, b), (c, d), (e, f); the other will be the required (arithmetical) set (a′, b′), (c′, d′), (e′, f′).

The process above described is perfectly general for all numbers $N = x^4 + y^4$. All Fermat's numbers (> 5) are of this form, $F_n \equiv 1^4 + E_{n-2}^4$: so that their algebraic partitions are

$$F_n = 1^2 + E_{n-1}^2 = G_{n-1}^2 + 2E_{n-2}^2 = F_{n-1}^2 - 2E_{n-2}^2.$$

Examples: $F_5 = 2^{32} + 1 = 641 \cdot Q_5$; $q_5 = 641$;

and $F_6 = 2^{64} + 1 = 274177 \cdot Q_6$; $q_6 = 274177$;

The results of the above process are shown in the table below. The algebraic partitions are shown first in the lines marked F_5, F_6; the partitions of q_5, q_6 were easily found (by trial); those of Q_5, Q_6 were found by conformal division (as above), and the second or (arithmetic) partitions, shown in the lines marked F'_5, F'_6 were found by conformal multiplication (as above).

$$F_5 = 1^2 + 65536^2 = 65535^2 + 2 \cdot 256^2 = 65537^2 - 2 \cdot 256^2,$$
$$q_5 = 25^2 + 4^2 = 21^2 + 2 \cdot 10^2 = 29^2 - 2 \cdot 10^2,$$
$$Q'_5 = 409^2 + 2556^2 = 2155^2 + 2 \cdot 1014^2 = 2973^2 - 2 \cdot 1084^2,$$
$$F'_5 = 20449^2 + 62264^2 = 24975^2 + 2 \cdot 42844^2 = 106897^2 - 2 \cdot 59716^2.$$

$$F_6 = 1^2 + 4294967216^2 = 4294967295^2 + 2 \cdot 65536^2,$$
$$q_6 = 89^2 + 516^2 = 523^2 + 2 \cdot 18^2,$$
$$Q_6 = 8083111^2 + 1394180^2 = 8192757^2 + 2 \cdot 282094^2,$$
$$F'_6 = 1438793759^2 + 4046803256^2 = 4274656527^2 + 2 \cdot 2950047882.$$

$$F_6 = 4294967297^2 - 2 \cdot 65536^2,$$
$$q_6 = 575^2 - 2 \cdot 168^2,$$
$$Q_6 = 9007423^2 - 2 \cdot 2631848^2,$$
$$F'_6 = 6063569153^2 - 2 \cdot 3026559664^2.$$

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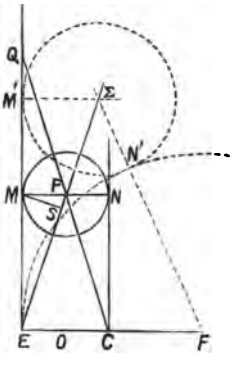
Note.—In the above cases the second 2-ic partitions have been found by help of the (previously known) factorizations. Conversely, if the second (or arithmetical) 2-ic partitions could be found independently, they would suffice to effect the factorizations. The next higher number $F_7 = (2^{128} + 1)$ is known to be composite; but the factors are not known. Unfortunately the difficulty of effecting the required 2-ic partitions of so large a number (39 figures) is at present probably insuperable.

16104. (R. F. Davis, M.A.)—If E, F are fixed points and a variable parabola is described passing through E, F and having EF as the normal at E, prove geometrically that the locus of Σ (defined by producing ES to Σ so that $ES \cdot E\Sigma = EF^2$) is another parabola having E for its vertex and F for its focus.

Solution by C. E. Youngman, M.A.

Draw ESP, the focal chord of the variable parabola, and EM, PM, the tangents at E and P. Then MS is perpendicular to EP and PM to EM and EM to EF; therefore the tangent PM and the chord EF are parallel, and PC (bisecting EF) must be a diameter, and will cut EM at Q, the pole of EF, making

$$CP = PQ.$$



Hence $PM = \frac{1}{2}EC$, and a circle (P) with radius PM will touch the two fixed lines EM and CN; it will also cut orthogonally the circle ESM on diameter EM. Now invert with centre E and radius EF. S becomes Σ; the circle ESM becomes $\Sigma M'$, perpendicular to EM; CN becomes the circle (F) with radius FE; and the circle (P) becomes a circle (Σ) touching EM′ and (F). This circle cuts $\Sigma M'$ and EP orthogonally; therefore its centre is Σ; therefore

$$F\Sigma = FE + \Sigma M',$$

and the parabola traced by Σ has focus F and vertex E.

Note that the varying directrix goes through M and is perpendicular to QC, and therefore to MO, O being the mid-point of EC; hence the envelope of the directrix is a parabola with focus O and vertex E.

The Proposer contributes the following two very brief Solutions to the problem:—

First Proof.— $EZ \cdot EL = ES \cdot E\Sigma = EF^2$,

$$EL/\Sigma L = EZ/Ze = 4EZ/EF = 4EF/EL;$$

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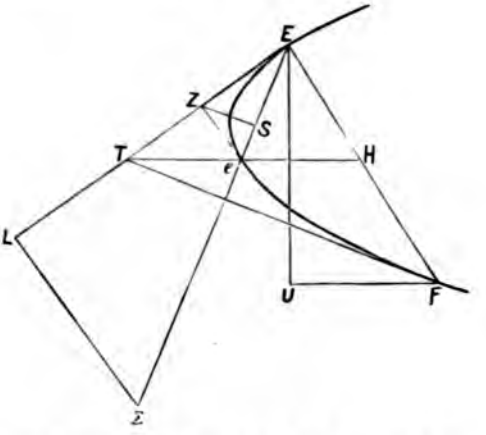
therefore $EL^2 = 4EF \cdot \Sigma L$;

therefore locus of Σ is a parabola vertex E, focus F.



Second Proof.—Since $ES \cdot E\Sigma = EF^2$, the triangles ESF, EFΣ are similar; therefore $SF/F\Sigma = ES/EF = EF/E\Sigma$; therefore

$$(SF-SE)/(F\Sigma-EF) = EF/E\Sigma = FU/\Sigma L,$$

since $$\angle UEH = \angle ETH = \angle SEZ.$$

But SF−SE = difference of perpendiculars from F, E on directrix
= FU;

therefore $$F\Sigma - EF = \Sigma L \quad \text{or} \quad F\Sigma = \Sigma L + EF.$$

Therefore the locus of Σ is a parabola focus F, and EL the tangent at its vertex.

16044. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—The vertex C of a triangle ABC moves on the arc ACB of a circle of which AB is a chord. Determine the conditions under whieh the sides a, b, c are expressible as integers, and give general formulæ for them.

Solution by the PROPOSER.

Here $$c^2 = a^2 + b^2 - 2ab\cos C = (a - b\cos C)^2 + (b\sin C)^2,$$

an ordinary Pythagorean case; the solution (a, b, c), of which, in integers, is given in terms of two arbitrary integers (t, u) by

$$c = t^2 + u^2, \quad a - b\cos C = t^2 - u^2, \quad b\sin C = 2tu.$$

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16083. (Professor SANJÁNA, M.A.)—Prove that the infinite product

$$\left(1\mp\frac{n^2}{1^2}\right)\left(1\mp\frac{n^2}{2^2}\right)\left(1\mp\frac{n^2}{4^2}\right)\left(1\mp\frac{n^2}{5^2}\right)\left(1\mp\frac{n^2}{7}\right)\left(1\mp\frac{n^2}{8^2}\right)\left(1\mp\frac{n^2}{10^2}\right)\ldots$$

$$= \tfrac{1}{3}\left(2\begin{matrix}\cos\\ \cosh\end{matrix}\tfrac{3}{2}\pi n+1\right).$$

Solutions by (I.) M. V. ARUNACHALAN, M.A.; (II.) *by* C. M. ROSS.

(I.) From the form for $\cos\theta$ as an infinite product, viz.,

$$\cos\theta = \prod_{n=1}^{n=\infty}[1-4\theta^2/(2n-1)^2\pi^2],$$

we easily deduce

$$(\cos\phi+\cos\theta)/(1+\cos\theta) = \Pi\left\{1-\phi^2/[(2n-1)\pi\pm\theta]^2\right\} \ldots\ldots\ldots(1)$$

(*vide* Chrystal, Part II., p. 359, Ex. 18). In (1) put $\phi = \frac{3}{2}n\pi$ and $\theta = \frac{1}{3}\pi$: we get

$$\tfrac{1}{3}(2\cos\tfrac{3}{2}\pi n+1) = \prod_{r=1}^{r=\infty}\left(1-\frac{\frac{1}{9}(4n^2)}{(2r-1\pm\frac{1}{3})^2}\right) = \prod_{r=1}^{r=\infty}\left(1-\frac{n^2}{(3n-\frac{1}{2})^2}\right)$$

$$= \left(1-\frac{n^2}{1^2}\right)\left(1-\frac{n^2}{2^2}\right)\left(1-\frac{n}{4^2}\right)\left(1-\frac{n^2}{5^2}\right)\ldots.$$

In (1) put $\phi = \frac{3}{2}in\pi$ and $\theta = \frac{1}{3}\pi$. Then we get

$$\tfrac{1}{3}(2\cosh\tfrac{3}{2}\pi n+1) = \left(1+\frac{n^2}{1^2}\right)\left(1+\frac{n^2}{2^2}\right)\left(1+\frac{n^2}{4^2}\right)\left(1+\frac{n^2}{5^2}\right)\ldots.$$

(II.) (i.) $\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2.\pi^2}\right)\left(1-\frac{\theta^2}{3^2.\pi^2}\right)\ldots.$

Put $\theta = \pi n$; then

$$\sin\pi n = \pi n\left(1-\frac{n^2}{1^2}\right)\left(1-\frac{n^2}{2^2}\right)\left(1-\frac{n^2}{3^2}\right)\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, put $\theta = \frac{1}{3}\pi n$; then

$$\sin\tfrac{1}{3}\pi n = \tfrac{1}{3}\pi n\left(1-\frac{n^2}{3^2.1^2}\right)\left(1-\frac{n^2}{3^2.2^2}\right)\left(1-\frac{n^2}{3^2.3^2}\right)\ldots\ldots\ldots(2).$$

Therefore (1)/(2) gives

$$\frac{\sin\pi n}{\sin\frac{1}{3}\pi n} = 3\left(1-\frac{n^2}{1^2}\right)\left(1-\frac{n^2}{2^2}\right)\left(1-\frac{n^2}{4^2}\right)\left(1-\frac{n^2}{5^2}\right)\left(1-\frac{n^2}{7^2}\right)\ldots;$$

therefore $\tfrac{1}{3}(2\cos\tfrac{3}{2}\pi n+1) = \left(1-\frac{n^2}{1^2}\right)\left(1-\frac{n^2}{2^2}\right)\left(1-\frac{n^2}{4^2}\right)\ldots,$

since $$\frac{\sin\pi n}{\sin\frac{1}{3}\pi n} = 3-4\sin^2\tfrac{1}{3}\pi n.$$

(ii.) $$\sinh\theta = \theta\left(1+\frac{\theta^2}{\pi^2}\right)\left(1+\frac{\theta^2}{2^2.\pi^2}\right)\left(1+\frac{\theta^2}{3^2.\pi^2}\right)\ldots.$$

Put $\theta = \pi n$, $\frac{1}{3}\pi n$ successively

$$\sinh\pi n = \pi n\left(1+\frac{n^2}{1^2}\right)\left(1+\frac{n^2}{2^2}\right)\left(1+\frac{n^2}{3^2}\right)\ldots,$$

$$\sinh\tfrac{1}{3}\pi n = \tfrac{1}{3}\pi n\left(1+\frac{n^2}{3^2.1^2}\right)\left(1+\frac{n^2}{3^2.2^2}\right)\left(1+\frac{n^2}{3^2.3^2}\right)\ldots;$$

therefore $$\frac{\sinh\pi n}{\sinh\frac{1}{3}\pi n} = 3\left(1+\frac{n^2}{1^2}\right)\left(1+\frac{n^2}{2^2}\right)\left(1+\frac{n^2}{4^2}\right)\ldots;$$

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therefore $\frac{1}{3}(2\cosh\frac{2}{3}\pi n+1)=\left(1+\frac{n^2}{1^2}\right)\left(1+\frac{n^2}{2^2}\right)\left(1+\frac{n^2}{4^2}\right)\ldots,$

since $$\frac{\sinh\pi n}{\sinh\frac{1}{3}\pi n}=3+4\sinh^2\tfrac{1}{3}\pi n.$$

Mr. A. M. NESBITT, M.A., gives the following brief and very neat Solution :—

We have $\left(1-\frac{m^2}{1^2}\right)\left(1-\frac{m^2}{2^2}\right)\left(1-\frac{m^2}{3^2}\right)\ldots$ ad inf. $=\sin m\pi/m\pi$.

Put $3m=n$: then the given expression when multiplied by $\sin\frac{1}{3}n\pi/\frac{1}{3}n\pi$ becomes $\sin n\pi/n\pi$. Thus the expression is equal to $\frac{1}{3}\sin n\pi/\sin\frac{1}{3}n\pi$, *i.e.*, $\frac{1}{3}(2\cos\frac{2}{3}n\pi+1)$.

Similarly the expression with lower signs is equal to $\frac{1}{3}\sinh n\pi/\sinh\frac{1}{3}n\pi$, *i.e.*, to $\frac{1}{3}(3+4\sinh^2\frac{1}{3}n\pi)$, which is $\frac{1}{3}(2\cosh\frac{2}{3}n\pi+1)$.

16112. (H. BATEMAN, B.A.)—Prove that (with the usual notation for the Legendre polynomial), if x, y, z lie between ±1,

$$\sum_0^\infty(2n+1)\,P_n(x)\,P_n(y)\,P_n(z)=\frac{2}{\pi\sqrt{(1-x^2-y^2-z^2+2xyz)}}\quad\text{or }0,$$

according as the quantity under the square root is positive or negative.

Solution by the Hon. G. R. DICK, M.A.

Denote for brevity the expression $1-x^2-y^2-z^2+2xyz$ by Δ, and let Ω be a function of z which $=1/\sqrt{\Delta}$ when Δ is positive, and $=0$ when Δ is negative. Suppose Ω exhibited as a series of sphere functions in z of the first kind, so that

$$\Omega=\sum_0^\infty a_nP_n(z)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

whence $$\frac{2}{2n+1}a_n=\int_{-1}^{1}\Omega P_n(z)\,dz.$$

Now $\Omega=0$, except when z lies between $xy-\sqrt{[(1-x^2)(1-y^2)]}$ and $xy+\sqrt{[(1-x^2)(1-y^2)]}$. Call these expressions z_0 and z_1 ; thus

$$a_n=\frac{2n+1}{2}\int_{z_0}^{z_1}\frac{P_n(z)\,dz}{\sqrt{\Delta}}.$$

Now put $z=xy-\sqrt{[(1-x^2)(1-y^2)]}\cos\phi$;

so that $dz=\sqrt{[(1-x^2)(1-y)]}\sin\phi\,d\phi$;

also $\Delta=(1-x^2)(1-y^2)\sin^2\phi$;

therefore

$$\int_{z_0}^{z_1}\frac{P_n(z)\,dz}{\sqrt{\Delta}}=\int_0^\pi P_n\left\{xy-\sqrt{[(1-x^2)(1-y^2)]}\cos\phi\right\}d\phi=\pi P_n(x)\,P_n(y),$$

a well known result. See Heine, *Handbuch der Kugelfunctionen*, Vol. I., p. 313.

When $x=1$, $y=1$ each side of (1) is zero, except when $z=1$, in which case each side is infinite.

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Note by the PROPOSER.

Conditions that a function of a real variable x can be expanded in a convergent series of Legendre polynomials with the above law for determining the coefficients are given by Darboux, "Approximation des Fonctions de grands Nombres," *Liouville's Journal* (3e série), t. IV., p. 393, 1878. The conditions are the same as those for a Fourier expansion, except that, if the function $f(x)$ becomes infinite at $x = \pm 1$, $(x \mp 1)^{\frac{1}{2}} f(x)$ must tend to zero as x approaches the singularity. The function given above clearly satisfies these conditions, and so the expansion is valid.

16065. (V. RAMASWAMI AIYAR.)—Let S denote the parabolic spiral $r = a\theta^2$. Let S′ denote the connected curve $r = a(\theta^2 - 1)$. Let S″ denote the negative pedal of S′. Prove (1) that any arc of the curve S″ is equal in length to the corresponding arc of its pedal S′; (2) that the radius of curvature at any point of S″ is equal to the distance of that point from the origin; (3) that the locus of the point of intersection of normals at corresponding points of S′ and S″ is a spiral of Archimedes; (4) that the evolute of S″ is an involute of a circle; and (5) that any arc of the curve S is equal in length to the corresponding arc of its second negative pedal.

Solutions (I.) *by* M. T. NARANIENGAR, M.A., *and* K. S. PATRACHAN; (II.) *by* M. V. ARUNACHALAN, M.A.

(I.) Let P′ be a point on S′ and P″ the corresponding point on S″, and let r', r'' denote OP′ and OP″; then

$$r' = a(\theta^2 - 1);$$

therefore

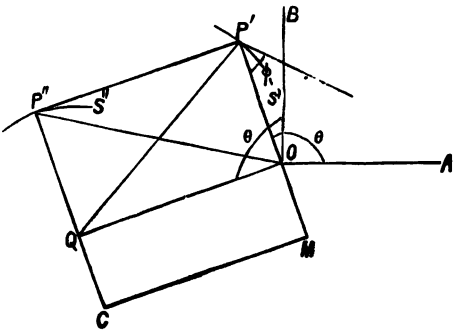
$$\cot\phi = 2\theta/(\theta^2 - 1)$$

and

$$\sin\phi = \frac{(\theta^2 - 1)}{(\theta^2 + 1)}.$$

Hence

$$r'' = r'/\sin\phi$$
$$= r'(\theta^2 + 1)/(\theta^2 - 1)$$
$$= r' + 2a,$$



ϕ being the angle between OP′ and the tangent at P′.

(1) Now, if s', s'' denote arcs of S′ and S″, $\cos\phi = dr'/ds' = dr''/ds''$. But $dr' = dr''$. Therefore $ds' = ds''$. Hence, by integration, any arc of S′ = corresponding arc of S″.

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(2) $r'' = r' + 2a = p'' + 2a$, if p'' denote the perpendicular on the tangent at P''; therefore

$$\rho \text{ of } S'' = r''(dr''/dp'') = r''.$$

(3) The normal at P' evidently bisects r''; so that, if Q be the intersection of the normals at P' and P'', $OQ = r' \cot\phi = 2a\theta$, *i.e.*, the locus of Q is a spiral of Archimedes with OB for initial line.

(4) If C be the centre of curvature at P'', complete the rectangle P'P''CM. Then $OM = 2a$ = constant; whence CM touches a circle; *i.e.*, the evolute of S'' is the involute of a circle.

(5) In the curve S or $r = a\theta^2$, $\cot\phi = 2/\theta$; therefore, if r_2 denote the radius vector of the second negative pedal S_2.

$$r_2 = r/\sin^2\phi = r(1 + \cot^2\phi) = r(1 + 4/\theta^2) = r + 4a:$$

so that $dr_2 = dr$. Hence $\cos\phi = dr/ds = dr_2/ds_2$; therefore $ds = ds_2$ and arc of S = the corresponding arc of S_2.

(II.) Let P be a point on S' and P' the corresponding point on S''. Now OP = the p of the point P' and its

$$\psi = \tfrac{1}{2}\pi + \theta;$$

therefore $d\psi = d\theta$. Now

$$p = OP = a(\theta^2 - 1) \ldots (a);$$

therefore

$$dp/d\psi = dp/d\theta = 2a\theta \ldots (1);$$

therefore

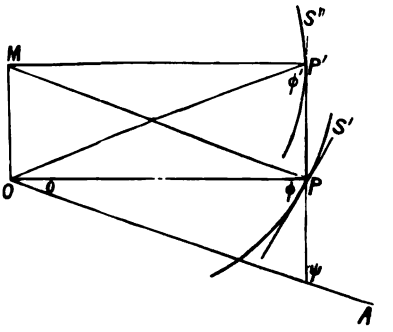
$$d^2p/d\psi^2 = 2a \ldots\ldots (2).$$

$PP' = t = dp/d\psi = 2a\theta$; therefore

$$OP'^2 = OP^2 + PP'^2 = a^2(\theta^2 + 1)^2;$$

therefore

$$OP' = a(\theta^2 + 1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$



The radius of curvature at the point P'

$$= \rho = p^4 + (d^2p/d\psi^2) = a(\theta^2 - 1) + 2a = a(\theta^2 + 1) = OP' \text{ by (3).}$$

This proves the second part of the question. Again,

$$ds'/d\theta = ds'/d\psi = \rho = a(\theta^2 + 1) \ldots\ldots\ldots\ldots\ldots (4)$$

where ds' is an element of arc of S'', and

$$ds/d\theta = \sqrt{[r^2 + (dr/d\theta)^2]} = a(\theta^2 + 1) \ldots\ldots\ldots\ldots\ldots (5)$$

where ds is an element of arc of S'. From (4) and (5) it follows that corresponding arcs of S' and S'' are equal. This proves the first part of the question.

Let M be the point of intersection of the normals at P and P'. Join OM. From (a) and (1) it follows that $\tan\phi' = OP/PP' = (\theta^2 - 1)/2\theta$ and $\tan\phi'$ is also equal to $r\,d\theta/dr = (\theta^2 - 1)/2\theta$; therefore $\phi' = \phi$; therefore $\angle OP'M = \angle OPM$; therefore $\angle MOP$ = a right angle. Hence OMPP' is a rectangle; therefore $OM = PP' = 2a\theta = 2a(\angle MOP - \frac{1}{2}\pi)$; therefore the locus of M is the spiral $r = 2a(\theta - \frac{1}{2}\pi)$, which proves Part 3 of the Question.

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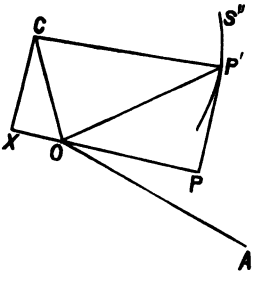
Let C be the centre of curvature at P'. Then P'C = OP' by the second part of the Question as proved above. Draw CX perpendicular to CP' and produce PO to meet CX in X. Then OX is obviously perpendicular to CX, and CX is the normal to the evolute of S'' at the point C. Now

$$OX = PX - PO = P'C - PO$$
$$= OP' - OP$$
$$= a(\theta^2+1) - a(\theta^2-1) = 2a.$$

Since $OX = 2a$, CX must touch the circle centre O and radius $2a$, which shows that the evolute of S'' is an involute of a circle.



Let P'' be a point on the second negative pedal of S corresponding to the point P on S. It is easily seen that the ϕ's of all the three curves are equal. Hence the triangles OPP' and OP'P'' are similar; therefore

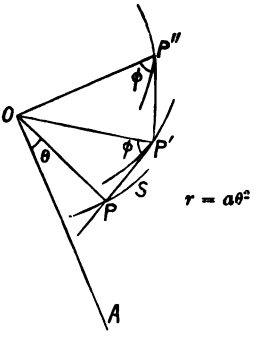
$$OP'^2 = OP \cdot OP'' \quad \ldots\ldots\ldots\ldots (1).$$

Now $OP'^2 = OP^2 + PP'^2 = a^2\theta^4 + (2a\theta)^2$

$$= a^2\theta^2(\theta^2+4);$$

therefore

$$OP'' = a(\theta^2+4) \quad \ldots\ldots\ldots\ldots (2).$$



Let ds be an element of arc of S and ds'' an element of arc of its second negative pedal, and let $\chi = \angle AOP''$.

$$d\chi = d\theta - 2d\phi.$$

But $\tan\phi = \frac{1}{2}\theta$; therefore

$$d\phi = 2d\theta/(\theta^2+4);$$

therefore

$$d\chi = \theta^2 d\theta/(\theta^2+4) \quad \ldots\ldots\ldots\ldots (3).$$

Now $ds^2 = dr^2 + r^2 d\theta^2 = (4a^2\theta^2 + a^2\theta^4)\,d\theta^2 = a^2\theta^2(\theta^2+4)\,d\theta^2 \quad \ldots\ldots\ldots\ldots (4),$

and $ds''^2 = dr''^2 + r''^2 d\chi^2$ (where $r'' = OP''$) $= 4a^2\theta^2 + a^2\theta^4 d\theta^2$

$$= a^2\theta^2(\theta^2+4)\,d\theta^2 \quad \ldots\ldots\ldots\ldots (5).$$

From (4) and (5) it follows that corresponding arcs of S and of its second negative pedal are equal.

15909. (H. L. Trachtenberg, B.A.)—ABC is a triangle; X is the point of contact with the side BC of the escribed circle corresponding to A. Show that an ellipse can be described with A, X as foci passing through B and C, and show that the lengths of the semi-major axis, semi-minor axis, and semi-latus rectum are respectively

$$\tfrac{1}{2}s, \quad \sqrt{[s(s-b)(s-c)/a]}, \quad 2(s-b)(s-c)/a$$

also eccentricity = AX/s.

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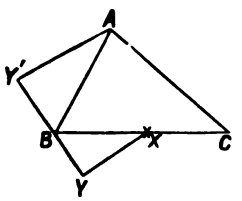
Solutions (I.) *by* A. M. NESBITT, M.A., *and* W. AUSTIN SLEIGH, B.A.; (II.) *by* SARADAKANTA GANGULI, M.A., F. W. REEVES, M.A., *and* R. J. WHITAKER.

(I.) Since $AB+BX = AC+CX = s$, the first part of the theorem is obvious, as also that s = major axis, and that eccentricity $= AX/s$. Let YBY′ bisect the exterior angle at B; then



$$XY = XB \sin XBY = (s-c)\cos \tfrac{1}{2}B,$$

and $$AY' = c \cos \tfrac{1}{2}B,$$

i.e., the square of the semi-minor axis is

$$\{[c(s-c)]\,[s(s-b)]\}/ac,$$

which agrees with the result propounded. Hence the semi-latus rectum, which is obtained by dividing this expression by $\frac{1}{2}s$, likewise takes the form stated.

(II.) Let the escribed circle touch AB and AC at Y and Z respectively. Then $AB+BX = AY = AZ = AC+CX$; therefore an ellipse with foci at A and X can be described through B and C. The semi-major axis $= \frac{1}{2}(AB+XB) = \frac{1}{2}AY = \frac{1}{2}s$,

$$\text{(minor-axis)}^2 = \text{(major-axis)}^2 - AX^2 = s^2 - [c^2 + (s-c)^2 - 2c(s-c)\cos B]$$
$$= 4s(s-b)(s-c)/a;$$

therefore semi-minor axis $= \sqrt{\{s(s-b)(s-c)/a\}}$,

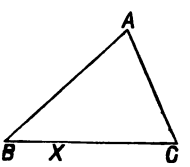
semi-latus rectum = (semi-minor axis)^2/semi-major axis
$= 2(s-b)(s-c)/a$,

and eccentricity $= AX/s$. This proposition suggests that, if the incircle touch BC at D, a hyperbola with foci at A and D can be described so as to pass through B and C.

Note.—The PROPOSER had intended the theorem on the hyperbola to be the subject of a separate question; it had been considered by him, and had furnished a series of results analogous to those of the given theorem.

The PROPOSER and Mr. C. M. Ross demonstrate the theorem as follows:—

Since BX and CX respectively $= s-c$ and $s-b$, therefore $BA+BX = s$ and $CA+CX = s$. Hence the ellipse with A, X for foci and $\frac{1}{2}s$ for semi-major axis passes through B and C. If l be its semi-latus rectum, since BXC is a focal



chord, $2/l = 1/BX + 1/CX = 1/(s-c) + 1/(s-b)$
$= a/[(s-b)(s-c)]$;

therefore $l = [2(s-b)(sc)]/a$;

the semi-minor axis $= \sqrt{(l \,.\, \text{semi-major axis})} = \sqrt{\{[s(s-b)(s-c)]/a\}}$.

the eccentricity = distance between foci / major axis = AX/s.

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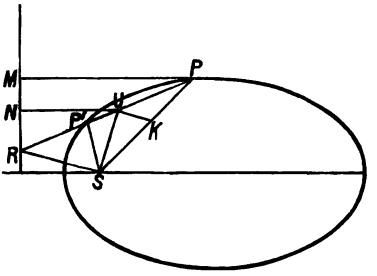
16070. ("CYMRO.")—Prove geometrically that a chord of an ellipse which subtends a constant angle at the focus always touches an ellipse with the same focus and directrix.

Solution by W. F. BEARD, M.A.

Let PP′ be a chord of an ellipse subtending a constant angle at the focus S, and let it meet the directrix at R. Bisect ∠PSP′ by SU, which is therefore perpendicular to SR. Draw PM, UN perpendicular to the directrix and UK perpendicular to SU. UK is parallel to SR; therefore SK/SP = PU/RP = UN/PM; therefore



$$SK/UN = SP/PM$$

therefore SK/UN is a constant ratio. Also SU/SK is a constant ratio, since ∠USK is constant; therefore SU/UN is a constant ratio, and the locus of U is an ellipse focus S and directrix MR. Also, since ∠USR is a right angle, UR is a tangent to this ellipse. Thus the envelope of PP′ is an ellipse focus S and directrix MR.

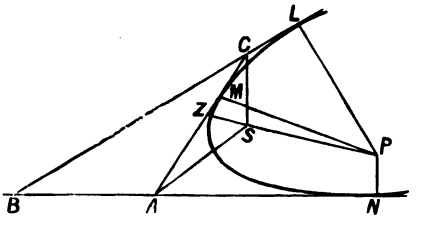
16103. (R. F. SCOTT, M.A.)—Normals PL, PM, PN are drawn to a parabola, focus S. Prove that the diameter of the circle circumscribing the triangle formed by the tangents to the parabola at the points L, M, N is equal to SP.

Solution by M. T. NARANIENGAR, M.A.

Let ABC be the triangle formed; then the circle ABC passes through S and its diameter

$d = (SA \cdot SC)/SZ$

where SZ = perpendicular on AC. Now, if SL, SM, SN be denoted by r_1, r_2, r_3, $SA^2 = r_2 r_3$ and $SC^2 = r_1 r_2$ and $SZ^2 = r_2 a$ (where $4a$ = latus rectum). Hence $d = \sqrt{(r_1 r_2 r_3/a)}$. But, in the case of concurrent normals $r_1 \cdot r_2 \cdot r_3 = a\text{SP}^2$. Therefore $d = SP$.



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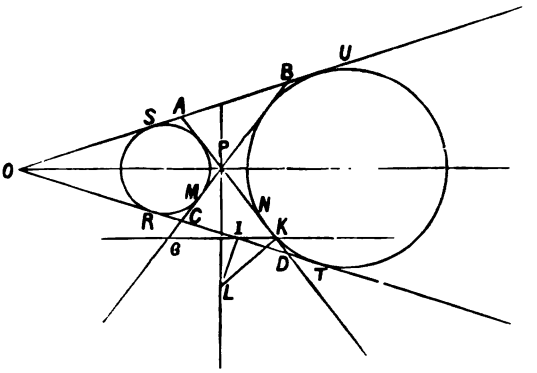
16127. (C. E. YOUNGMAN, M.A.)—**Construct the two circles of which there are given the two centres of similitude, the length of one common tangent, and the direction of another.**

Solution by HENRY RIDDELL, M.E.

Let O and P be the two centres of similitude, and let the direction of AD and hence of BC be given. Then RT, the common tangent,

$$= BC = PC + PD.$$



The problem therefore reduces to the following:—Through O, a point on the external bisector of the vertical angle, draw a transversal OD, cutting off PC + PD = a given length. Draw PL, the internal bisector, and cut off PK = ½RT. Draw KL perpendicular to PK. Then is L a point on the circum-circle of the required triangle. Describe on OL as diameter a circle; then the middle point of CD lies on this circle, since PL is the bisector of the angle at P. But the middle point also lies on KG, parallel to OP, and is therefore the intersection of these two loci I. Draw OI and OA, and the circles may be described at once. If the cross tangent be chosen as the given length, a similar problem is obtained, but varied by the given point being placed on the internal bisector and the difference of sides being known.

16074. (Professor LAUVERNAY.)—**Inscrire dans un cercle de rayon R un triangle de surface maximum, sachant que ses médianes se rencontrent en un point G distant du centre O d'une longueur GO = a.**

Solution by W. F. BEARD, M.A.

Let the sides of the triangle be a, b, c; Δ its area. $abc = 4R\Delta$; therefore abc must be a maximum. Also $OG^2 = R^2 - \frac{1}{9}\Sigma a^2$ and $OG = a$; therefore $\Sigma a^2 = 9(R^2 - a^2) = \text{const.}$ Hence $a^2b^2c^2$ or abc is a maximum when $a = b = c$. Thus $a^2 = 3(R^2 - a^2)$ ....

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15981. (Professor NEUBERG.)—Etant donné un paraboloï e

$$z = ax^2 + by^2,$$

on mène un plan tangent quelconque α, et par la trace de α sur le plan xOy on mène un plan β perpendiculaire à α. Trouver l'enveloppe du plan β.

Solution by J. HODGKINSON.

The tangent plane at any point (x_1, y_1, z_1) of the paraboloid $z = ax^2 + by^2$ is $z + z_1 = 2axx_1 + 2byy_1$. The plane through the trace on xOy is

$$2axx_1 + 2byy_1 = kz + z_1.$$

These planes are perpendicular if $k + 4a^2x_1^2 + 4b^2y_1^2 = 0$; *i.e.*, the plane is $2axx_1 + 2byy_1 + z(4a^2x_1^2 + 4b^2y_1^2) - (ax_1^2 + by_1^2) = 0$,

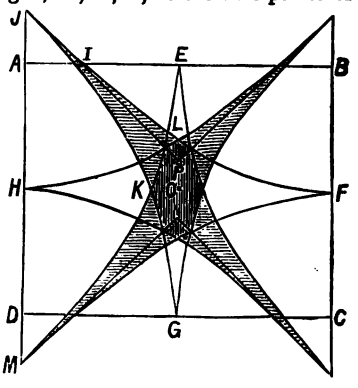
the envelope of which we find in the usual way to be the cubic surface

$$ax^2 + by^2 = 4abz(x^2 + y^2).$$

16084. (D. BIDDLE.)—Define the area within which lies the centre of a circle that cuts all four sides of a given rectangle (*a*) twice, (*b*) at least once.

Solution by JAMES BLAIKIE, M.A.

Let ABCD be the given rectangle; E, F, G, H the mid-points of its sides. Let eight parabolas be described, each of which has one of the vertices of the rectangle as focus and one of the opposite sides as directrix. It is clear, from the properties of the parabola, that, if X be a point within the area AEKH, then XA, the largest radius which would give a circle meeting AD in two points, would be less than the perpendicular from X on BC. Thus the required area (*a*) is the vertically shaded part left when AEKH and seven similar parts of parabolic areas have been removed from the rectangle.



To find the area (*b*), we observe that the common tangents drawn to the parabolas JKG, ILF meet them in I and J, where the parabolas cut AB and AD produced. In addition to the area already found, the areas IPL, JPK clearly satisfy the condition (*b*). Thus he horizontally shaded area, as well as the vertically shaded, satisfies (*b*).

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16117. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Show that

$$n^{2n} \equiv 1 \pmod{p}$$

where $p = x^4 + y^4 = 256n + 1$ (a prime) and $y = 4\eta$.

Solution by the PROPOSER.

It has been shown in the Proposer's paper "On 2 as a 16-ic Residue" in *Proc. Lond. Math. Soc.*, Vol. XXVII., Art. 16, that $2^{(p-1)/16} \equiv +1$ when $p = x^4 + y^4$ (a prime), if $y = 4\eta$. Here $2^8 n \equiv -1 \pmod{p}$; therefore

$$(2^8)^{(p-1)/128} n^{(p-1)/128} \equiv (-1)^{(p-1)/128} \pmod{p},$$

whence $\quad 2^{(p-1)/16} n^{2n} \equiv +1 \quad$ and $\quad n^{2n} \equiv +1 \pmod{p}$.

16076. (A. M. NESBITT, M.A.)—Factorize the determinant whose first row is $a_2 + a_3 - 2a_1, \quad a_2^2 + a_3^2 + a_2a_3, \quad a_1(a_1^2 + 2a_2a_3)$.

Solution by M. V. ARUNACHALAN, M.A.

Let $$D \equiv \begin{vmatrix} a_2 + a_3 - 2a_1, & a_2^2 + a_3^2 + a_2a_3, & a_1^3 + 2a_1a_2a_3 \\ a_3 + a_1 - 2a_2, & a_3^2 + a_1^2 + a_3a_1, & a_2^3 + 2a_1a_2a_3 \\ a_1 + a_2 - 2a_3, & a_1^2 + a_2^2 + a_1a_2, & a_3^3 + 2a_1a_2a_3 \end{vmatrix}.$$

Subtracting the last row from each of the first two rows,

$$D = (a_3 - a_1)(a_3 - a_2) \begin{vmatrix} 3, & a_1 + a_2 + a_3, & -(a_1^2 + a_3^2 + a_1a_3) \\ 3, & a_1 + a_2 + a_3, & -(a_2^2 + a_3^2 + a_2a_3) \\ a_1 + a_2 - 2a_3, & a_1^2 + a_2^2 + a_1a_2, & a_3^3 + 2a_1a_2a_3 \end{vmatrix}.$$

Subtract the second row from the first. Then

$$D = (a_1 - a_3)(a_2 - a_3)(a_3 - a_1)(a_1 + a_2 + a_3) \times \begin{vmatrix} 0, & 0, & 1 \\ 3, & \Sigma a_1, & -(a_2^2 + a_3^2 + a_2a_3) \\ a_1 + a_2 - 2a_3, & a_1^2 + a_2^2 + a_1a_2, & a_3^3 + 2a_1a_2a_3 \end{vmatrix}$$

$$= (a_1 - a_3)(a_2 - a_3)(a_3 - a_1)(a_1 + a_2 + a_3) \begin{vmatrix} 3, & a_1 + a_2 + a_3 \\ a_1 + a_2 - 2a_3, & a_1^2 + a_2^2 + a_1a_2 \end{vmatrix}$$

$$= (a_1 - a_2)(a_2 - a_3)(a_3 - a_1)(a_1 + a_2 + a_3)(2\Sigma a_1^2 + \Sigma a_2a_3).$$

16055. (Professor SANJÁNA, M.A.)—S and H are two given points, P and Q two other given points, all in one plane. Draw through P and Q two straight lines mutually perpendicular to cut SH in U and V, so that (SUHV) may be harmonic.

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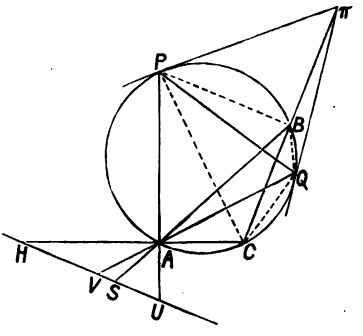
Extension by the PROPOSER.

Taking the more general case when the two lines through P and Q contain a given angle, we may proceed thus. On PQ describe a circle, so that the segment PAQ contains the given angle, and find π, the pole of PQ. Inscribe the triangle ABC so that AB, AC go through S, H, and BC goes through π. Then, producing PA, QA, we get the range (SUHV) harmonic. For PBQC is a harmonic quadrilateral; hence A (PBQC) is a harmonic pencil, *i.e.* (USVH) or (SUHV) is a harmonic range. In Mr. Riddell's Solution to the Question as set, BC, being perpendicular to PQ (the diameter), is parallel to the tangents at P and Q, *i.e.*, goes through the pole of PQ which is at infinity. It is evident that there are two solutions when the angle PAQ is not right.



16049. (Professor NANSON.)—Find the condition that the three pairs of meets of three conics S_1, S_2, S_3 with three lines L_1, L_2, L_3 respectively may lie on a conic.

Solution by W. F. BEARD, M.A.

Let $S_1 \equiv F_1(x, y), \ldots$. Let L_1, L_2, L_3 meet at A, B, C, and let the co-ordinates of A, B, C be $x_1y_1, \ldots$. Let L_1 meet S_1 at D_1E_1; $\ldots$. By Carnot's theorem, if D_1E_1, D_2E_2, D_3E_3 lie on a conic, then

$$[(BD_1 . BE_1)/(CD_1 . CE_1)][(CD_2 . CE_2)/(AD_2 . AE_2)] \times [(AD_3 . AE_3)/(CD_3 . CE_3)] = 1.$$

But $[(BD_1 . BE_1)/(CD_1 . CE_1)] = [F_1(x_2y_2)]/[F_1(x_3y_3)], \ldots$.

Thus the required condition is

$$F_1(x_2y_2)\, F_2(x_3y_3)\, F_3(x_1y_1) = F_1(x_3y_3)\, F_2(x_1y_1)\, F_3(x_2y_2).$$

16116. (H. L. TRACHTENBERG, B.A.)—If the capitals are the minors of the corresponding small letters in $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, prove that, if

$$F/f = G/g = H/h,$$

then

$$(A/f + H/g + G/h)/(1/f) = (H/f + B/g + F/h)/(1/g) = (G/f + F/g + C/h)/(1/h).$$

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Solution by C. M. Ross.

$$A = bc-f^2, \quad B = ca-g^2, \quad C = ab-h^2,$$
$$F = gh-af, \quad G = hf-bg, \quad H = fg-ch.$$

Let $F/f = G/g = H/h = \lambda$;

i.e., $(gh-af)/f = (hf-bg)/g = (fg-ch)/h = \lambda$.

Then $(A/f + H/g + G/h)/(1/f) = bc-f^2 + f(fg-ch)/g + f(hf-bg)/h$
$= [gh(bc-f^2) + hf(fg-ch) + fg(hf-bg)]/gh$
$= (bcgh-cfh^2+f^2gh-bg^2f)/gh$
$= (fg-ch)(hf-bg)/gh = \lambda^2$.

Similarly each of other fractions $= \lambda^2$. Therefore, &c.

11024. (H. J. Woodall.)—By the division of a periodic continued fraction by a whole number, another periodic continued fraction results, whose period is a simple number of times as long as the period of the original continued fraction.

Remark by Professor Sanjána, M.A.

Every periodic continued fraction may be represented by the surd $(a+\sqrt{b})/c$. When $a = 0$, at least, the proposition is not true. For instance, $\sqrt{(46)}$ has a period of 12 repeated quotients; while $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$ of this surd have periods of 12, 8, 12, 15, and 8 respectively. When $a \neq 0$ I believe a more complex relation will hold between the periods than is contemplated by the Question.

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1 110. (Suggested by and leading to a Solution of 10581. James Blaikie, M.A.)—Trace from geometrical considerations the locus of a point P which moves so that two given straight lines AB, CD subtend the same angle at the point, considering the cases where ABCD is (1) a square, (2) a rectangle, (3) a parallelogram, (4) an irregular quadrilateral.

Solution by the Proposer.

(1) When ABCD is a square, as in Fig. 1, it is clear that the produced parts of the diagonals, the straight line joining I, J, the mid-points of AD, BC, and the arcs AD, BC of the circum-circle are the locus.

(2) When ABCD is a rectangle, as in Fig. 2, we have still the arcs AB, CD and the line IJ as parts of the locus. To find the other part we take as centre points E, F, equidistant from AB, CD on their per-

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pendicular bisector, and describe circles ABPQ, CPQD, meeting in P, Q, points on the locus. If N be the mid-point of EF, we have

$$PN^2 = r^2 - a^2 \quad \text{and} \quad ON^2 = r^2 - b^2,$$

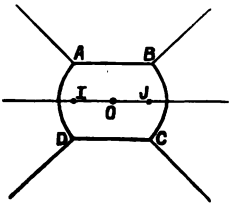

FIG. 1.

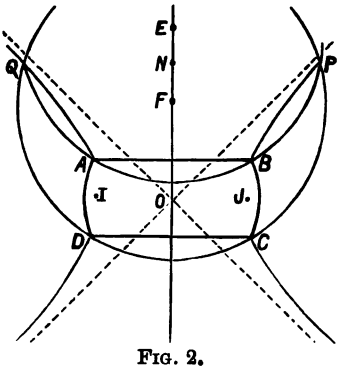

FIG. 2.

where r is the radius of the circle and $2a$, $2b$ are the sides of the rectangle; and in the limit where a and b may be neglected in comparison with r PN = ON; which shows that the curve has two asymptotes which pass through O and are parallel to the bisectors of the angles of the rectangle.

(3) When ABCD is a parallelogram, as in Fig. 3, we find by a similar construction that the locus has four infinite branches starting from

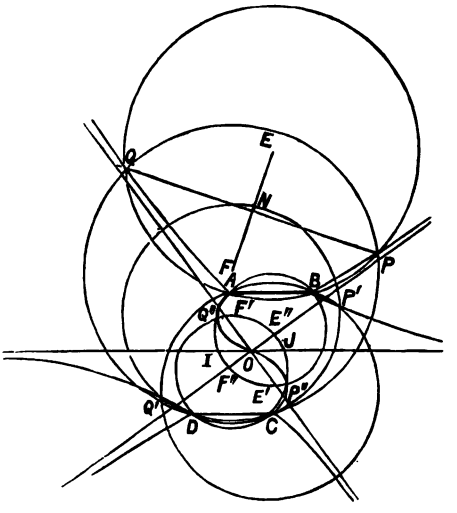

FIG. 3.

A, B, C, D, whose asymptotes go through the centroid O and are parallel to the bisectors of the angles of the parallelogram. The

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straight line IJ and the arcs AD, BC are replaced by two curves BP′ and DQ′ having IJ as asymptotes, and the S-shaped curve AQ″OP″. The three pairs of circles with centres E, F; E′, F′; E′, F″ show how the points P, Q; P′, Q′; P″, Q″ are obtained.

(4) When ABCD is an irregular quadrilateral, as in Fig. 4, the forms obtained for the locus vary considerably according to the shape of ABCD. We may find points on the locus by describing pairs of circles on AB, CD as chords with radii proportional to AB, CD. The three points of intersection of the opposite sides and diagonals of the quadrilateral are clearly points on the locus, and other points may be found by simple constructions, but it is difficult to trace the curve completely. The straight lines IJ, KL, joining the mid-points of AD, BC; AC, BD, are asymptotes; for the projections of AB, CD perpendicular to these lines are equal and the angles which they subtend at infinitely distant points are equal. One curious point is that the pairs of infinite branches appear to approach IJ and KL from the same sides, and not from opposite sides, as is usual.

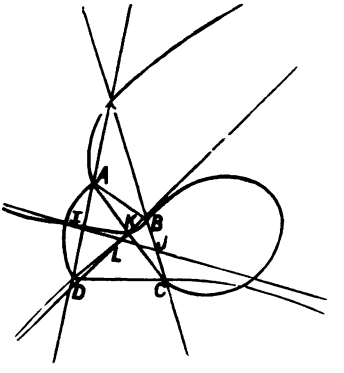

FIG. 4.

16082. (R. CHARTRES.)—Sum the series

$$1-\frac{C_1}{3}+\frac{C_2}{6}-\frac{C_3}{10}+\frac{C_4}{15}-\ldots,$$

where $C_r \equiv$ combinations of n things r together.

Another Solution by R. J. WHITAKER.

Let S represent the series. Then

$$\frac{(n+1)(n+2)}{2}S$$

$$\equiv \frac{(n+2)(n+1)}{2!}-\frac{(n+2)(n+1)n}{3!}+\frac{(n+2)(n+1)n(n-1)}{4!}-\ldots$$

$$= (1-1)^{n+2}-[1-(n+2)] = n+1;$$

therefore $$S = \frac{2(n+1)}{(n+1)(n+2)} = \frac{2}{n+2},$$

the sum required.

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NOTE. *Extension of the above.*—The sum of the series

$$1-\frac{C_1}{a_1}+\frac{C_2}{a_2}-\frac{C_3}{a_3}+\dots$$

where $C_r \equiv$ number of combinations of n things r together, and $a_r \equiv$ coefficient of x^r in the expansion of $(1-x)^{-m}$, m being a positive integer, is $(m-1)/(n+m-1)$.

The PROPOSER adds that the original problem may at once be solved as follows :—

The series is simply twice the difference of the known series :

$$\frac{1}{m+1}-\frac{n}{1}\frac{1}{m+2}+\frac{n(n-1)}{2!}\frac{1}{m+3}-\dots=\frac{n!}{(m+1)(m+2)\dots(m+n+1)}$$

when $m = 0$ and $m = 1$.

16081. (Professor LAUVERNAY.)—Déterminer p et q de manière que les racines de l'équation $x^4+pax^3+(m^2+m)a^2x^2+qa^3x+a^4=0$ soient en progression géométrique, et trouver ces racines en fonction de a et de m.

Solutions (I.) *by* R. F. DAVIS, M.A., A. M. NESBITT, M.A., *and* C. M. ROSS ; (II.) *by* W. F. BEARD, M.A.

(I.) Assume the roots are $a\lambda^3$, $a\lambda$, $a\lambda^{-1}$, $a\lambda^{-3}$. Then the equation

$$[x^2-(\lambda^3+\lambda^{-3})ax+a^2][x^2-(\lambda+\lambda^{-1})ax+a^2]=0$$

must be identical with the given equation. Equating like coefficients,

$$\lambda^3+\lambda+\lambda^{-1}+\lambda^{-3}=-p=-q, \qquad \lambda^4+\lambda^2+\lambda^{-2}+\lambda^{-4}+2=m^2+m.$$

Therefore $\lambda^2+\lambda^{-2}=m$ is one solution evidently ; or $\lambda+\lambda^{-1}=\sqrt{(m+2)}$. Thus

$$-p=-q=m\sqrt{(m+2)} \quad \text{and} \quad \lambda=\tfrac{1}{2}[\sqrt{(m+2)}+\sqrt{(m-2)}].$$

Note that the ambiguity of the radical sign is to be considered in the preceding.

(II.) Let a, ar, ar^2, ar^3 be the roots in geometric progression. Then

$$a(1+r+r^2+r^3)=-pa, \quad ra^2(1+r+2r^2+r^3+r^4)=(m^2+m)a^2\dots(1,\ 2),$$
$$r^3a^3(1+r+r^2+r^3)=-qa^3, \quad r^6a^4=a^4 \dots\dots\dots\dots(3,\ 4).$$

From (1) and (3), $r^3a^2=qa^2/p$, and, from (4),

$$r^3a^2=\pm a^2 ; \quad \text{therefore} \quad p=\pm q \dots\dots\dots\dots(5).$$

Take the positive sign in each case ; then, from (2),

$$(1/r^2)(1+r+2r^2+r^3+r^4)=m^2+m ;$$

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therefore $(r+1/r)^2+(r+1/r)-m(m+1)=0$,

$$r+1/r=m \quad \text{or} \quad -(m+1) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (6).$$

Taking the first solution,

$$r=\tfrac{1}{2}[m\pm\sqrt{(m^2-4)}]=\{\tfrac{1}{2}[\sqrt{(m+2)}\pm\sqrt{(m-2)}]\}^2;$$

therefore $r^{\frac{1}{2}}=\lambda\pm\mu$,

where $\lambda=\tfrac{1}{2}\sqrt{(m+2)}$, $\mu=\tfrac{1}{2}\sqrt{(m-2)}$ and $\lambda^2-\mu^2=1 \ldots \ldots \ldots \ldots (7)$;

therefore $r^{-\frac{1}{2}}=\lambda\mp\mu$.

Again, taking the upper sign, we have, from (4), $a=a(\lambda-\mu)^3$. Thus the roots are

$$a(\lambda-\mu)^3, \quad a(\lambda-\mu)^3(\lambda+\mu)^2, \quad a(\lambda-\mu)^3(\lambda+\mu)^4, \quad a(\lambda-\mu)^3(\lambda+\mu)^6,$$

or $$a(\lambda-\mu)^3, \quad a(\lambda-\mu), \quad a(\lambda+\mu), \quad a(\lambda+\mu)^3,$$

from (7). We should get the same roots by taking the other value of $r^{\frac{1}{2}}$.

From (1), we have

$$-pa=a[(\lambda-\mu)^3+(\lambda+\mu)^3+\lambda-\mu+\lambda+\mu],$$

$$p=-2\lambda(\lambda^2+3\mu^2+1)=\tfrac{1}{4}\{\sqrt{(m+2)}[m+2+3(m-2)+4]\}$$

$$=-m\sqrt{(m+2)}=q.$$

If, instead of m, we put $-m-1$, we obtain different values for p, q, and the four roots.

If we take the negative sign in (5), the values of r will be given by

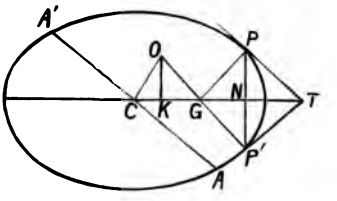
$$(r+1/r)^2+(r+1/r)+m^2+m=0,$$

and the values of r will be very complicated.

16087. (C. E. Youngman, M.A.)—Prove that similar ellipses having a common diameter envelop a pair of circles.

Solution by M. T. Naraniengar, M.A.

Let one of the similar ellipses through A, A′ be drawn. Draw PT, the parallel tangent, and PNP′, the double ordinate. Then obviously TP′ and AA′ are equally inclined to the axis; and therefore the circle through APA′ touches the ellipse at P. Now, for similar ellipses PG : CA is constant. Therefore, if O be the centre of this circle, CO : CK = PG : NG; therefore



CO : PG ≡ CK : NG = CG : 2NG = constant for similar ellipses,

and PG is constant; therefore CO is constant. That is, the circle is fixed.

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Note by the PROPOSER.

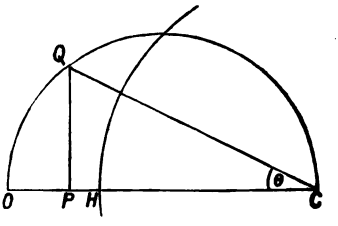
The Question is really no more than a corollary to Prop. 5 in Mr. W. F. Beard's article on the equiconjugates (Vol. IX., p. 34, New Series) or to the following:—The chord of points on an ellipse whose eccentric angles are complementary is parallel to an equiconjugate. Another proof, working more directly from the envelope idea, may be given:—

Let P′ be an intersection (other than A, A′) of two consecutive ellipses: regarding A and P′ as points belonging to the first ellipse, find a and p', their homologues in the second (similar) ellipse. The double point of the figures is C; therefore $\angle CP'p' = \angle CAa$; therefore, ultimately, when P′ is a point on the envelope, the angle between radius and tangent at P′ is equal to that at A, and of the same sense. Hence CP′ is not the conjugate of CA, but the image of that conjugate in one of the axes (as above), and the circle AP′A′ touches both ellipse and envelope at P′. It follows that this circle is fixed and is itself the envelope; for, if not, whatever was the locus of P′ would be also the envelope of a series of coaxal circles AA′. And, as CP′ is not constant, C is not the centre of the circle: so that symmetry requires another equal circle to complete the envelope.

16075. (W. J. DOBBS, M.A.)—If we neglect the resistance of the atmosphere, a small body, placed initially at rest relatively to the Earth at a height of 500 miles above its surface, would fall to the ground in about 7½ minutes. Find how far the body falls and the velocity it acquires in the first 5 minutes.

Solution by the PROPOSER.

Let C be the centre of the Earth, O the point from which the body is let fall, H the point where it strikes the Earth's surface. Let R denote the radius of the Earth and h the distance OH, both in miles, and let v ml./sec. be the velocity at P after falling x miles. The acceleration due to gravity on the Earth's surface may be taken as



$$32{\cdot}2 \text{ ft./sec.}^2 = 0{\cdot}00610 \text{ ml./sec.}^2 = g \text{ (say)}.$$

The acceleration at P is $gR^2/(R+h-x)^2$. Hence

$$v\,dv/dx = gR^2/(R+h-x)^2.$$

Integrating this equation, we have

$$v^2 = 2gR^2x/[(R+h)(R+h-x)].$$

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Taking R as 4000 miles, so that $R = 8h$, we have

$$v = \tfrac{1}{2}\sqrt{[gR\,(x/h)\,(9-x/h)]}\,;$$

i.e.,
$$dx/dt = v = 6{\cdot}60\sqrt{[(x/h)/(9-x/h)]}.$$

For the purpose of integrating this equation, draw PQ, the ordinate of the semi-circle on OC as diameter, and let $\angle OCQ = \theta$. Then $x = (R+h)\sin^2\theta$; so that $x/h = 9\sin^2\theta$, whence $18h\cos^2\theta\,d\theta = 6{\cdot}60\,dt$; so that

$$t = \frac{9h}{6{\cdot}60}(\theta + \sin\theta\cos\theta)$$

$$= \frac{h}{6{\cdot}60}\left[9\sin^{-1}\tfrac{1}{3}\sqrt{\frac{x}{h}} + \sqrt{\left\{\left(9-\frac{x}{h}\right)\frac{x}{h}\right\}}\right]$$

$$= 75{\cdot}8\left[9\sin^{-1}\tfrac{1}{3}\sqrt{\frac{x}{500}} + \sqrt{\left\{\left(9-\frac{x}{500}\right)\frac{x}{500}\right\}}\right].$$

From this we obtain the following values :—

x	150	200	250	300	miles
t	248	286	319	349	sec.

and, by means of a graph, $x = 220$ when $t = 300$. Also, substituting 220 for x in the formula

$$v = 6{\cdot}60\sqrt{\left\{\frac{x}{500}\Big/\left(9-\frac{x}{500}\right)\right\}},$$

we have $v = 1{\cdot}49$ ml./sec.

16096. (T. STUART, D.Sc.)—Find integral values of x and y satisfying the equations (i.) $2x^4 - y^2 = 256999$, (ii.) $2x^2 - y^4 = 356962$.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

(i.) $N_1 = 256999 = 233 \cdot 1103 = (19^2-2\cdot 8^2)(41^2-2\cdot 17^2)$
$= (19\cdot 41 \pm 16\cdot 17)^2 - 2(19\cdot 17 \mp 8\cdot 41)^2 = 507^2 - 2\cdot 5^2 = \ldots.$

Also $7^2 - 2\cdot 5^2 = -1$; therefore

$N_1 = -(507^2-2\cdot 5^2)(7^2-2\cdot 5^2) = -[(507\cdot 7 \mp 2\cdot 5\cdot 5)^2 - 2(507\cdot 5 \mp 5\cdot 7)^2]$
$= -(3499^2 - 2\cdot 2500^2)\ldots;$

therefore
$$N_1 = 2\cdot 50^4 - 3499^2.$$

(ii.)
$$N_2 = 356962 = 2\cdot 178481 = 2\cdot p.$$

Here
$$178481 = 391^2 + 160^2 = 57^2 + 2\cdot 296^2.$$

From these data p can be expressed in the form $p = e^2 - 2f^2$ or $2f'^2 - e'^2$ by a method fully explained in the writer's paper on "Connexion of Quadratic Forms" in *Proc. Lond. Math. Soc.*, Vol. XXVIII., pp. 302 *et seq.* Thereby $p = 2\cdot 305^2 - 87^2$, whence

$$N_2 = 2p = 610^2 - 2\cdot 87^2.$$

Hence $N_2 = -(610^2 - 2\cdot 87^2)(1^2 - 2\cdot 1^2) = (610 \mp 2\cdot 87)^2 - 2(610 \mp 87)^2$
$= -(784^2 - 2\cdot 697^2),$

whence
$$N_2 = 2\cdot 697^2 - 28^4.$$

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16103. (R. F. Scott, M.A.)—Normals PL, PM, PN are drawn to a parabola, focus S. Prove that the diameter of the circle circumscribing the triangle formed by the tangents to the parabola at the points L, M, N is equal to SP.

Analytical Solution by A. M. Nesbitt, M.A.

The tangents at L, M, N being denoted by $y = x/l + al$, ..., those at M, N will intersect in amn, $a(m+n)$. Calling this point A, we shall at once have for the length of BC $a(m-n)\sqrt{(1+l^2)}$, while

$$\tan BAC = (m \sim n)/(1+mn);$$

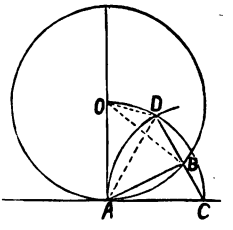
so that $$\sin^2 A = (m-n)/\sqrt{[(1+m^2)(1+n^2)]}.$$

Thus diameter of circum-circle $= a\sqrt{[(1+l^2)(1+m^2)(1+n^2)]}$, and this, as l, m, n satisfy $am^3 + (2a-x)m - y = 0$ (in which x, y are co-ordinates of P), reduces at once to $\sqrt{[y^2 + (2a-x)^2 - 2a(2a-x) + a^2]}$, *i.e.*, to $\sqrt{[y^2 + (a-x)^2]}$; which roves the theorem.

To inscribe a Regular Heptagon in a given Circle, centre O, radius = r.

By Lindley Latham.

Draw a diameter, and through one extremity A draw a line AC perpendicular to it. With centre A and radius AO, describe an arc ODC, cutting AC in C. With centre C and same radius, describe another arc, cutting the first in D. Join CD, and let CD cut the given circle in B. AB will be one side of the inscribed heptagon. Join OB, OD, and AD. Then



$$AC = CD = AD = AO = OB = r.$$

ACD is an equilateral triangle: therefore

$$\angle CAD = \angle ADC = 60°.$$

Also $$\angle DAO = \angle CAO - \angle CAD = 90° - 60° = 30°.$$

Since AO = AD, $\angle AOD = \angle ADO$; therefore

$$2\angle ADO = 180° - \angle OAD = 180° - 30° = 150°;$$

therefore

$$\angle ADO = 75° \quad \text{and} \quad \angle ODB = \angle ODA + \angle ADB = 75° + 60° = 135°.$$

Now $\frac{1}{2}OD = AD \cos ADO = r\cos 75°$; therefore

$$OD = 2r\cos 75° = 2r\,[(\sqrt{3}-1)/(2\sqrt{2})] = [(\sqrt{3}-1)/\sqrt{2}]r.$$

In the triangle OBD, $OB = r$, $OD = [(\sqrt{3}-1)/\sqrt{2}]r$, $\angle ODB = 135°$; $\sin OBD/OD = \sin ODB/OB$; therefore

$$\sin OBD = (\sin ODB/OB)\,OD = (1/\sqrt{2})\,[(\sqrt{3}-1)/\sqrt{2}]\,r\,(1/r)$$
$$= \tfrac{1}{2}(\sqrt{3}-1) = \tfrac{1}{2}(1{\cdot}7320508 - 1)$$
$$= {\cdot}3660254 = \sin 21° 28' 14{\cdot}52'' \text{ or } \sin 158° 31' 45{\cdot}48'';$$

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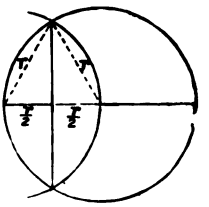
but $158° 31' 45{\cdot}48'' + 135° > 180°$; therefore the value $158° 31' 45{\cdot}48''$ is inadmissible; therefore $\angle OBD = 21° 28' 14{\cdot}52''$.

$$\angle DOB = 180° - (\angle ODB + \angle OBD) = 180° - (135° + 21° 28' 14{\cdot}52'')$$
$$= 180° - 156° 28' 14{\cdot}52'' = 23° 31' 45{\cdot}48''.$$

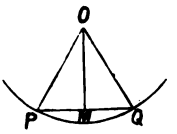
Now $\angle AOB = \angle AOD - \angle DOB = 75° - 23° 31' 45{\cdot}48'' = 51° 28' 14{\cdot}52''$ or $2' 31{\cdot}7''$ larger than the true angle at the centre.

$$[\tfrac{1}{7}(360°) = 51° 25' 42{\cdot}86''].$$



The above method therefore gives a result closer to the truth than the method of inscribing a regular heptagon in a given circle, by making its side equal to half that of the inscribed equilateral triangle; or, which is the same thing, equal to the median of an equilateral triangle having its sides equal to the radius of the circle; that is to say, equal to

$$\sqrt{[r^2 - (\tfrac{1}{2}r)^2]} = \sqrt{\tfrac{1}{4}(4r^2 - r^2)} = \tfrac{1}{2}\sqrt{3}\,r.$$



For let POQ be an isosceles triangle obtained by joining the ends of one side PQ of the heptagon inscribed by this method to the centre of the circle O, and let OM be drawn perpendicular to PQ. Then $PM = \tfrac{1}{2}PQ = \tfrac{1}{2} \cdot \tfrac{1}{2}\sqrt{3}\,r = \tfrac{1}{4}\sqrt{3}\,r$,

and, if $\angle POQ = \theta$, $\angle POM = \tfrac{1}{2}\theta$.

Now $\sin \tfrac{1}{2}\theta = MP/OP = \tfrac{1}{4}\sqrt{3}\,r\,(1/r) = \tfrac{1}{4}\sqrt{3}$

$$= \tfrac{1}{4}(1{\cdot}7320508) = {\cdot}4330127 = \sin 25° 39' 32{\cdot}06'';$$

therefore $\tfrac{1}{2}\theta = 25° 39' 32{\cdot}06''$;

therefore $\theta = 51° 19' 4{\cdot}12''$

or $6' 38{\cdot}74''$ smaller than the true angle at the centre.

[*Note.*—The above piece of work reaches us through the Author's father, from whom we learn that it is the investigation of a lad of fourteen. This gives it a special interest.—EDITOR.]

16139. (M. V. ARUNACHALAN, M.A.)—If in the expression $\dfrac{1}{1-x+x^2}$ the sum of the even terms be $\phi(x)$ and the sum of the odd terms be $\psi(x)$, then $[\phi(x)]^2 - [\psi(x)]^2 = \phi(-x^2) + \psi(-x^2)$.

[*Note.*—The Solver has anticipated the changes in the Question asked for by the Proposer, viz., of "expression" into *expansion of*, and of "terms" into *power terms*.—ED.]

Solution by W. TELFER *and* A. N. NESBITT, M.A.

Suppose $\dfrac{1}{1-x+x^2}$ is capable of expansion in the form

$$A_0 + A_1x + A_2x^2 + \ldots + A_rx^r + \ldots .$$

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Obviously $\frac{1}{1+x+x^2}$ will equal

$$A_0 - A_1x + A_2x^2 - \ldots + (-1)^r A_r x^r.$$

Thus $2\phi(x) \equiv 2A_0 + 2A_2x^2 + \ldots = \frac{1}{1-x+x^2} + \frac{1}{1+x+x^2} = \frac{2(1+x^2)}{1+x^2+x^4}$,

and $2\psi(x) = 2A_1x + 2A_3x^3 + \ldots = \frac{1}{1-x+x^2} - \frac{1}{1+x+x^2} = \frac{2x}{1+x^2+x^4}$.

But obviously

$$\frac{1}{1+x^2+x^4} = A_0 + A_1(-x^2) + A_2(-x^2)^2 + \ldots = \phi(-x^2) + \psi(-x^2);$$

therefore we have $[\phi(x)]^2 - [\psi(x)]^2 = \phi(-x^2) + \psi(-x^2)$.

10550. (E. M. Langley, M.A.)—Deduce from Ptolemy's theorem the existence of Simson's line.

Solution by Rev. C. O'Connell.

$Lb = x\,BZ - z\,BX$ (Ptol. th.);

therefore $L = \frac{x\,BZ}{b} - \frac{z\,BX}{b}$.

So $M = \frac{y\,AZ}{a} + \frac{z\,AY}{a}$.

From similar triangles

$$\frac{BX}{b} = \frac{AY}{a};$$

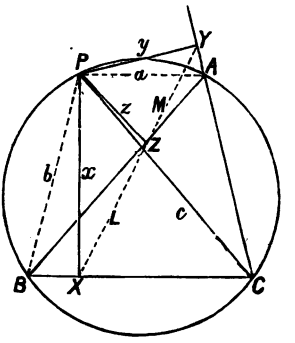
therefore

$$L + M = \frac{x\,BZ}{b} + \frac{y\,AZ}{a} = \frac{x\,CY}{c} + \frac{y\,CX}{c}.$$

But $x\,CY + y\,CX = XY\,c$ (Ptol. th.);

therefore $L + M = \frac{XY\,c}{c} = XY$;

therefore X, Y, Z are collinear.



16115. (Hon. G. R. Dick.)—Show by pure algebra (without importing any geometrical terms or notions) that the homogeneous equation of the second degree in x, y, z,

$$\begin{vmatrix} x^2, & x_1^2, & x_2^2, & x_3^2, & x_4^2, & a \\ y^2, & y_1^2, & y_2^2, & y_3^2, & y_4^2, & b \\ z^2, & z_1^2, & z_2^2, & z_3^2, & z_4^2, & c \\ yz, & y_1z_1, & y_2z_2, & y_3z_3, & y_4z_4, & f \\ zx, & z_1x_1, & z_2x_2, & z_3x_3, & z_4x_4, & g \\ xy, & x_1y_1, & x_2y_2, & x_3y_3, & x_4y_4, & h \end{vmatrix} = 0$$

(wherein $x_1, y_1, z_1, \ldots$ are given, and $a, \ldots, h$ arbitrary constants), which

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appears to involve five arbitrary parameters, in reality involves only one. [The obvious geometrical analogue is:—If four points of a conic are given, one more condition is sufficient to determine the conic completely.]

Solution by Professor NANSON *and* A. M. NESBITT, M.A.

Let A, B, ... be the co-factors of a, b, ... in the given equation, which is therefore $aA + bB + cC + fF + gG + hH = 0$.
We then have identically

$$(ABCFGH)(x_r y_r z_r)^2 = 0$$

where $r = 1, 2, 3, 4$. From these four equations we can solve linearly for four of the co-factors in terms of another two—say, P, Q—and so express the given equation in the form $pP + qQ = 0$, where p, q are linear in a, b, c, f, g, h. Thus the given equation really involves one parameter only, viz., p/q.

The only case in which the argument used fails is when the 15 four-line determinants formable from the array

$$\begin{array}{cccccc} x_1^2, & y_1^2, & z_1^2, & y_1z_1, & z_1x_1, & x_1y_1, \\ x_2^2, & y_2^2, & z_2^2, & y_2z_2, & z_2x_2, & x_2y_2, \\ x_3^2, & y_3^2, & z_3^2, & y_3z_3, & z_3x_3, & x_3y_3, \\ x_4^2, & y_4^2, & z_4^2, & y_4z_4, & z_4x_4, & x_4y_4 \end{array}$$

all vanish; and in this case the given equation is identically true.

15790. (Professor SANJÁNA, M.A.)—Resolve into prime factors (or at least into six factors) the numbers $6^{42} + 1$ and $24^{30} + 1$.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

(i.) $$N = 6^{42} + 1 = \left(\frac{6^{42}+1}{6^{14}+1}\ \frac{6^2+1}{6^6+1}\right)\frac{6^{14}+1}{6^2+1}\ \frac{6^6+1}{6^2+1}(6^2+1)$$

$$= (N'_3/N_3)\,N_7N_3N_1 \text{ (suppose)};$$

$$N_1 = 6^2 + 1 = 37; \quad N_3 = (6^6+1)/(6^2+1) = 13 \cdot 97:$$

$$N_7 = (6^{14}+1)(6^2+1) = 421 \cdot 5030761.$$

The two factors $N'_3 = (6^{42}+1)/(6^{14}+1)$, $N_3 = (6^6+1)/(6^2+1)$ are both of form (styled Sext-Aurifeuillian) $N_{vi} = (Y^6+1)/(Y^2+1)$, where $Y = 6y^2$. All such numbers are (algebraically) resolvable into two factors (say L, M), where

$N_{vi} = L, M$; $L = P - Q$, $M = P + Q$; $P = 1 + 3Y + Y^2$, $Q = 6y(1+Y)$.
Taking $Y = 6^7$ gives $y = 6^3$; $N'_3 = [(6^7)^6+1]/[(6^7)^2+1]$,

$P' = 1 + 3 \cdot 6^7 + 6^{14} = 78365003905$, $Q' = 6 \cdot 6^3(1+6^7) = 362798352$,

$L' = 78002205553$, $M' = 78727802257$;

and, since N'_3 is divisible by N_3, it follows that L'_3, M'_3 must be divisible by the factors of N_3. On actual trial it is found that

$$L'_3 = 97 \cdot 804146449; \quad M'_3 = 13 \cdot 6055984789.$$

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Hence $N'_3/N_3 = \frac{36^{21}+1}{36^7+1}\frac{36+1}{36^3+1}$, the algebraic prime factor of $(36^{21}+1)$.

Now the writer has compiled a table showing the roots of the congruence $(x^7+1)/(x+1) \equiv 0 \pmod p$ up to $p \not> 10{,}000$. This table shows that N'_3/N_3 contains no prime factors $< 10{,}000$, and on applying Lawrence's process for the determination of high primes which are factors of (a^n+1) it is found that the two large numbers $L'_3/97$, $M'_3/13$ are both prime. Hence

$$6^{42}+1 = (804146449 \cdot 6055984789)(421 \cdot 5030761)(13 \cdot 97)(37);$$

(ii.) $$N = 24^{30}+1 = \left(\frac{24^{30}+1}{24^{10}+1}\frac{24^2+1}{24^6+1}\right)\frac{24^{10}+1}{24^2+1}\frac{24^6+1}{24^2+1}(24^2+1)$$

$$= (N'_3/N_3)\,N_5N_3N_1 \text{ (suppose)},$$

$$N_1 = 24^2+1 = 577; \quad N_3 = (24^6+1)/(24^2+1) = 349\,(13 \cdot 73);$$

$$N_5 = (24^{10}+1)/(24^2+1) = 61 \cdot 1801385941.$$

As to N_5, the writer has compiled a table of solutions of the congruence $(y^{10}+1)/(y^2+1) \equiv 0 \pmod p$ up to $p \not> 10{,}000$; this shows that 61 is a divisor of N_5, and that there are no more divisors $< 10{,}000$. On applying Lawrence's process for determining the high prime factors of (a^n+1) it is found that the large factor of N_5 is prime.

Next, the two factors

$$N'_3 = (24^{30}+1)/(24^{10}+1), \quad N_3 = (24^6+1)/(24^6+1)$$

are both of form (styled Sext-Aurifeuillian) $N_{vi} = (Y^6+1)/(Y^2+1)$ where $Y = 6y^2$. All such numbers are resolvable into two factors (say L, M) by the formulæ given above under (i.).

Taking $Y = 24 = 6 \cdot 2^2$ gives $y = 2$, $N_3 = L_3M_3$, where $L_3 = 349$ and $M_3 = 13 \cdot 73$ (as above).

Taking $Y' = 24^5$ gives $y' = 6^2 \cdot 2^5$, $N'_3 = L'_3M'_3$,

$$P'_3 = 1 + 3 \cdot 24^5 + 24^{10} = 63403404853249,$$

$$Q'_3 = 6 \cdot 6^2 \cdot 2^5(1+24^5) = \tfrac{1}{2} \cdot 24^3(1+24^5) = 55037664000,$$

$$L'_3 = 63348367189249, \quad M'_3 = 63458442517249,$$

and, since N'_3 is divisible by N_3, it follows L'_3, M'_3 must be divisible by L_3, M_3 or by M_3, L_3 respectively. On actual trial it is found that

$$L'_3/L_3 = 181513946101, \quad M'_3/M_3 = 66868748701.$$

Now the writer has compiled a table showing the roots of the congruence $(y^{15}+1)(y+1)/(y^5+1)(y^3+1) \equiv 0 \pmod p$ up to $p \not> 10{,}000$; this shows that N'_3/N_3 contains no prime factors $< 10{,}000$; but, with the means at present available, it would be very laborious to attempt further resolution of the two large factors L'_3/L_3, M'_3/M_3. Thus, finally, N has been resolved into 8 factors of which 6 are prime.

The following discussion of the Question is due to Mr. A. H. Bell:—

Firstly, $$6^{42}+1 = (x^2+1)(x^{12}-x^{10}+x^8-x^6+x^4-x^2+1)$$
$$\times [(x^{14}+3x^7+1)+x^4(x^7+1)]\,[(x^{14}+3x^7+1)-x^4(x^7+1)],$$

in which $6 = x$ and x^4 is derived from $6x^7 = x^8$.

Again, $$24^{30}+1 = (x^2+1)(x^8-x^6+x^4-x^2+1)$$
$$\times [(x^{10}+3x^5+1)+12x^2(x^5+1)]\,[(x^{10}+3x^5+1)-12x^2(x^5+1)].$$

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In this case $24 = x$ and $(12x^3)^2 = 6x^6$. $N^{6n} + 1 = (x^{2n}+1)(x^{4n} - x^{2n} + 1)$. From this last general factor, two can be found when $6x^n$ is a square in $(x^{2n} + 3x^n + 1)^2 - 6x^n(x^n + 1)^2$. [For this latter transformation we are indebted to the Proposer.]

16119. (R. F. Davis, M.A.)—Eliminate λ, κ from the three equations $(\lambda^2+1)x - \lambda\kappa = a$, $(\lambda^2+1)y + \kappa = \lambda^2 a$, $(\lambda^2+1)(x^2+y^2) = \kappa^2$ (i., ii., iii.).

Solutions by (I.) Professor Sanjána, M.A.; (II.) *by* C. M. Ross *and* W. F. Beard, M.A.

(I.) From (i.) $x = (a + \lambda\kappa)/(\lambda^2+1)$, from (ii.) $y = (\lambda^2 a - \kappa)/(\lambda^2+1)$; hence, from (iii.), $\kappa^2/(\lambda^2+1) = [(a+\lambda\kappa)^2 + (\lambda^2 a - \kappa)^2]/(\lambda^2+1)^2$;

i.e., $\kappa^2(\lambda^2+1) = (a+\lambda\kappa)^2 + (\lambda^2 a - \kappa)^2$

whence $\kappa = a(\lambda^4+1)/2(\lambda^2-\lambda)$.

Hence $x(\lambda^2+1) = a + a(\lambda^4+1)/2(\lambda-1)$

and $y(\lambda^2+1) = \lambda^2 a - a(\lambda^4+1)/2(\lambda^2-\lambda)$;

these give $a\lambda^4 - 2\lambda^3 x + 2\lambda^2 x + 2\lambda(a-x) + (2x-a) = 0$

and $(2y-a)\lambda^4 + 2\lambda^3(a-y) + 2\lambda^2 y - 2\lambda y + a = 0$.

From these two equations the eliminant may be found by Sylvester's or Bezout's method.

(II.) Adding (i.) and (ii.), $(\lambda^2+1)(x+y) - (\lambda-1)\kappa = a(\lambda^2+1)$; therefore $\kappa = (\lambda^2+1)(x+y-a)/(\lambda-1)$ ..... (iv.).

From (i.) and (ii.), $(\lambda^2+1)(x+\lambda y) = a(\lambda^2+1)$ ..... (v.);

(iv.) and (iii.) give $(\lambda^2+1)(x^2+y^2) = (\lambda^2+1)^2(x+y-a)^2/(\lambda-1)^2$; therefore $(\lambda-1)^2(x^2+y^2) = (\lambda^2+1)(x+y-a)^2$;

therefore $(\lambda-1)^2/(\lambda^2+1) = (x+y-a)^2/(x^2+y^2)$;

therefore $(-2\lambda)/(\lambda^2+1) = [2xy - 2a(x+y) + a^2]/(x^2+y^2)$.

Put $M = x^2+y^2$, $N = 2a(x+y) - 2xy - a^2$; then $(\lambda^2+1)/2\lambda = M/N$;

therefore $(\lambda+1)^2/(\lambda-1)^2 = (M+N)/(M-N)$; therefore

$$\lambda/1 = [\sqrt{(M+N)} + \sqrt{(M-N)}]/[\sqrt{(M+N)} - \sqrt{(M-N)}]$$
$$= [M + \sqrt{(M^2-N^2)}]/N.$$

(v.) becomes $(\lambda^2+1)(x+\lambda y) = a(\lambda+1)(\lambda^2+1-\lambda)$;

therefore $2M\lambda/N = (x+\lambda y) = a(\lambda+1)(2M\lambda/N - \lambda)$

therefore $2M(x+\lambda y) = a(\lambda+1)(2M-N)$;

therefore $\lambda[2My - a(2M-N)] = a(2M-N) - 2Mx$;

therefore $[M + \sqrt{(M^2-N^2)}][2My - a(2M-N)] = N[(a2M-N) - 2Mx]$;

therefore

$$(M^2-N^2)[2My - a(2M-N)]^2 = \{N[a(2M-N) - 2Mx] - M[2My - a(2M-N)]\}^2;$$

therefore

$$(M^2-N^2)[2My - a(2M-N)]^2 = [a(2M-N)(M+N) - 2M(My + Nx)]^2,$$

the eliminant.

N.B.—The values of M and N may now be restored; but the result is very long.

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16069. (H. L. TRACHTENBERG, B.A.)—The common points of two conics are on a circle. Prove that the circle on the line joining the points of contact of a common tangent of the conics as diameter cuts this circle orthogonally.

Solutions (I.) *by* R. F. DAVIS, M.A., *and others;* (II.) *by* W. F. BEARD, M.A.

(I.) Take the common tangent as axis of x and a perpendicular through its middle point O as the axis of y. Then the equations of the two conics are

$S \equiv (x-a)^2 + 2hxy + by^2 + 2fy = 0, \quad S' \equiv (x+a)^2 + 2h'xy + b'y^2 + 2f'y = 0$

where $2a$ is the length of the common tangent.

Now, every conic through the four points of intersection is of the form $S + \lambda S' = 0$. By hypothesis, one such conic is a circle: hence it must be possible to determine λ so that the coefficient of xy vanishes at the same time as the coefficients of x^2 and y^2 are equal. The circle must therefore be

$$h'[(x-a)^2 + 2hxy + by^2 + 2fy] = h[(x+a)^2 + 2h'xy + b'y^2 + 2f'y],$$

with the condition $h'-h = h'b-hb'$. Thus the equation of the circle becomes $x^2+y^2+a^2+$ terms in $x, y = 0$ and the tangent to it from $O = a$.

(II.) Let $u\alpha^2 + v\beta^2 + w\gamma^2 = 0, \quad u'\alpha^2 + v'\beta^2 + w'\gamma^2 = 0$ ..... (1, 2)

be the equations of two conics touching $l\alpha \pm m\beta \pm n\gamma = 0$. Then

$$\Sigma l^2/u = 0, \quad \Sigma l^2/u_1 = 0 \text{ .....} \quad (3).$$

It is easy to find that the points of contact with $l\alpha + m\beta + n\gamma = 0$ are $l/u, m/v, n/w$; $l/u', m/v', n/w'$. Any conic through the intersection of (1) and (2) is $(u + \kappa u')\alpha^2 + \ldots = 0$ .....(4).

If this is a circle, it must be the same as

$$a\beta\gamma + \ldots + (\lambda\alpha + \mu\beta + \nu\gamma)(a\alpha + b\beta + c\gamma) = 0;$$

therefore $b\nu + c\mu + a = 0, \ldots$ or $\mu/b + \nu/c = -a/bc, \ldots,$

from which

$$2\lambda/a = a/bc - b/ca - c/ab = (a^2 - b^2 - c^2)/abc = -2\cos A/a;$$

therefore $\lambda = -\cos A$, and the equation of the circle is

$$\Sigma a \cos A\alpha^2 = 0 \text{ .....} \quad (5).$$

Also, by comparing with (4), $(u + \kappa u')/a\cos A = (v + \kappa v')/b\cos B$ ...,

and, from (3), $[l^2(u + \kappa u')]/uu' + [m^2(v + \kappa v')]/vv' + \ldots = 0;$

hence

$$\Sigma a \cos A l^2/uu' = 0 \text{ .....} \quad (6).$$

Now the circle on $l/u, m/v, n/w$; $l/u', \ldots$ as diameter cuts (5) orthogonally if the polar of one of the points with respect to (5) passes through the other point. The polar of $l/u, \ldots$ is $l/u\, a\cos A\alpha + \ldots = 0$, and this passes through $l/u', \ldots$ if $\Sigma(l^2/uu')a\cos A = 0$, which is true from (6).

The PROPOSER and Mr. H. RIDDELL, M.E., solve the problem as follows:—

Let the circle through the common points of the conics be S_1, and

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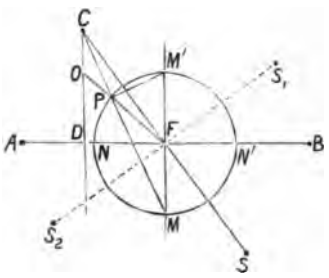
that on T_1T_2 as diameter, S_2; T_1, T_2 being the points of contact of the common tangent. Let T_1T_2 cut S_1 in BC. Then, since the two conics and S_1 have four common points, T_1T_2 cuts them in involution; therefore T_1, T_2 are the double points of this involution, and, if O is the mid-point of T_1T_2, $OB.OC = OT_1^2$. Thus B and C are inverse points with regard to S_2. Thus S_1, since it passes through them, cuts S_2 orthogonally.

16121. (Professor NEUBERG.)—On considère les hyperboles équilatères qui passent par deux points fixes A, B et dont un axe de symétrie passe par un point donné C. Trouver le lieu des centres.

Solutions (I.) *by* C. E. YOUNGMAN, M.A.; (II.) *by* C. M. ROSS.

(I.) Take F the mid-point of AB, and draw FM and CD perpendicular to AB. Let P be one of the centres, and PN, PN′ the corresponding asymptotes, cutting AB at N, N′; then AN = BN′; therefore FN = FN′, and F is the centre of the circle PNN′. Draw this circle, cutting FM at M and M′; then PM, PM′ are the axes of the hyperbola, and one of them, say PM, is to go through C. Let FP meet CD at O; then, since



FP = FM, OP = OC;

that is, we can find P by taking any point O on CD and setting off along OF a length OP equal to OC. The locus of P is therefore a strophoid with node at C and double focus at F.

We might regard PM, PM′ as the asymptotes of a rectangular hyperbola cutting FM at fixed points equidistant from F; thus getting a solution of Question 15371 (see *Reprint*, New Series, Vol. VII., p. 101)

On CF take FS = FC; then SM′ is parallel to CM, and therefore perpendicular to PM′; hence PM′ touches the parabola with focus S and directrix CD. The strophoid is the pedal of this parabola with respect to C. By turning the triangle CPM′ round C through half a right angle, and making it shrink at the same time to half its present area, we can bring PM′ into line with PN or PN′; hence these lines also envelop parabolas, whose foci are corners of the square on S as diagonal. And a further result may be put as follows:—Let two equal parabolas be so placed that their directrices meet at right angles at C, the direct double point of the two figures, and let a right angle NPN′ slide with one arm resting on each parabola; then the locus of P is a strophoid.

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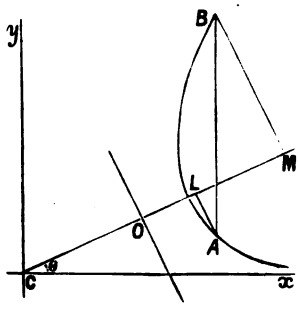
(II.) Take C as origin and axes parallel and perpendicular to AB. Let A be (h, k_1), B be (h, k_2). If CO be an axis of the equilateral hyperbola

$$AL^2-LO^2 = BM^2-MO^2 \quad ...(1),$$

let the co-ordinates of O be (X, Y). The equation of CM is

$$Yx-Xy = 0.$$



Therefore

$$AL = (Yh-Xk_1)/(X^2+Y^2)^{\frac{1}{2}},$$
$$AO^2 = (X-h)^2+(Y-k_1)^2;$$

therefore

$$AL^2-LO^2 = 2AL^2-AO^2$$
$$= [2(Yh-Xk_1)^2]/(X^2+Y^2)-(X-h)^2-(Y-k_1)^2,$$
$$BM^2-MO^2 = 2BM^2-BO^2$$
$$= [2(Yh-Xk_2)^2]/(X^2+Y^2)-(X-h)^2-(Y-k_2)^2;$$

therefore, substituting in (1),

$$2[(Yh-Xk_1)^2-(Yh-Xk_2)^2] = (X^2+Y^2)[(Y-k_1)^2-(Y-k_2)^2];$$

therefore $(k_1+k_2)Y^2+4XYh-(k_1+k_2)X^2 = 2Y(X^2+Y^2)$,

a cubic, the locus of the centres.

[N.B.—For the choice of axes the Solver is indebted to Mr. H. W. Richmond, Fellow of King's College, Cambridge.]

16132. (A. M. Nesbitt, M.A.)—Factorize (A, B, C being angles of a triangle)

$$\begin{vmatrix} 0, & x\cos C+y, & x\cos B+z \\ y\cos C+x, & 0, & y\cos A+z \\ z\cos B+x, & z\cos A+y, & 0 \end{vmatrix}.$$

Solutions (I.) *by the* Proposer *and* W. Telfer; (II.) *by* C. M. Ross.

(I.) Multiplying the columns respectively by sin A, sin B, sin C, and adding for a new first column, we can take out $x\sin A+y\sin B+z\sin C$; the remaining factor is at once seen to be $\sin A\,\Sigma yz\sin A$, and, as in our first operation we have multiplied the determinant by sin A, we see that the factors are $\Sigma x\sin A\,\Sigma yz\sin A$.

This might also be seen *ab initio* by multiplying the several rows by $yz\sin A$, $zx\sin B$, $xy\sin C$ respectively, and adding.

(II.) Let Δ denote the determinant; then

$$\Pi(\sin A)\,\Delta = \begin{vmatrix} 0, & x\sin B\cos C+y\sin B, & x\sin C\cos B+z\sin C \\ y\sin A\cos C+x\sin A, & 0, & y\sin C\cos A+z\sin C \\ z\sin A\cos B+x\sin A, & z\sin B\cos A+y\sin B, & 0 \end{vmatrix};$$

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therefore

$$\sin A\Delta = (x\sin A + y\sin B + z\sin C)\begin{vmatrix} 1, & x\cos C + y, & x\cos B + z \\ 1, & 0, & y\cos A + z \\ 1, & z\cos A + y, & 0 \end{vmatrix}$$

(by adding columns)

$$= (x\sin A + y\sin B + z\sin C)\begin{vmatrix} 1, & x\cos C + y, & x\cos B + z \\ 0, & -(x\cos C + y), & -(x\cos B - y\cos A) \\ 0, & z\cos A - x\cos C, & -(z + x\cos B) \end{vmatrix}$$

$$= (x\sin A + y\sin B + z\sin C)\,[xy(\cos A\cos C + \cos B) + yz\sin^2 A + zx(\cos A\cos B + \cos C)]$$

$$= (x\sin A + y\sin B + z\sin C)(xy\sin C\sin A + yz\sin^2 A + zx\sin A\sin B).$$

Therefore

$$\Delta = xyz\,(x\sin A + y\sin B + z\sin C)(x^{-1}\sin A + y^{-1}\sin B + z^{-1}\sin C).$$

16185. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Discuss the possibility of expression of a given number N in one or more of the four forms $(t^2 \sim 2u^4)$, $(t^4 \sim 2u^2)$. [All symbols integers; t prime to u.]

Solution by the PROPOSER.

A preliminary (necessary) condition that a number N may be expressible in one (or more) of the forms $(t^2 \sim 2u^4)$, $(t^4 \sim 2u^2)$ is that N must be expressible in one of the forms $(e^2 \sim 2f^2)$ [and note that either of them involves the other]. It seems possible that this is also a sufficient condition, as the writer finds that every prime of form $(e^2 \sim 2f^2) < 500$ is expressible in one of the required forms (most of them in two or three such), with the possible exception of 281 (in which the author has failed; no small values of t, u suffice).

The following is a general method of finding t, u (when possible). Express N in one of the forms $(e^2 \sim 2f^2)$, and let e, f be the minimum values in such forms. Let τ, v be any solution of the unit-forms $\tau^2 - 2v^2 = \pm 1$. Then

$$\pm N = (e^2 - 2f^2)(\tau^2 - 2v^2) = (e\tau \mathbin{\dot{\sim}} 2fv)^2 - 2(ev \mathbin{\dot{\sim}} f\tau)^2$$

where both signs are $-$ or both $+$.

If any one of the required forms be possible, then values of τ, v must exist giving $\quad e\tau \mathbin{\dot{\sim}} 2fv = \square \quad$ or $\quad ev \mathbin{\dot{\sim}} f\tau = \square$.

There seems to be no mode of selecting τ, v other than by trial of all successive solutions (τ_1, v_1), (τ_2, v_2), ... of the unit form in turn. [A table of these solutions up to τ, v each > 5 million will be found on p. 261 of the writer's *Tables of Quadratic Partitions*, 1904.]

As examples of numbers algebraically expressible in some of the required forms

$$N_1 = (x^4 + y^4) = 2(x^2 \pm xy + y^2)^2 - (x \pm y)^4 \quad \ldots\ldots\ldots\ldots (1),$$

$$N_2 = \tfrac{1}{2}(x^4 + y^4) = t^4 - 2[\tfrac{1}{2}(x \pm y)^2]^2, \text{ if } x^2 \pm xy + y^2 = t^2 \quad \ldots (2),$$

$$N_3 = (x^8 + y^8) = (x^4 + y^4)^2 - 2(xy)^4 \quad \ldots\ldots\ldots\ldots (3),$$

$$N_4 = \tfrac{1}{2}(x^8 + y^8) = 2[\tfrac{1}{2}(x^4 + y^4)]^2 - (xy)^4 \quad \ldots\ldots\ldots\ldots (4).$$

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A number N is sometimes algebraically expressible in more than one of the required forms, thus

$$N_5 = t^4 \sim 2u^4 = t^4 \sim 2(u^2)^2 = (t^2)^2 \sim 2u^4 \quad \ldots\ldots(5),$$

$$N_6 = (2u^2 \pm 1)^2 - 2u^4 = 2(u^2 \pm 1)^2 - 1^4 \quad \ldots\ldots(6),$$

$$N_7 = (u^2 \pm 2)^2 - 2 \cdot 1^4 = 2(u^2 \pm 1)^2 - u^4 \quad \ldots\ldots(7),$$

$$N_8 = t^4 - 2u_1^2 = 2u_4^2 - t^4 \text{ when } t^2 = u_1^2 + u_4^2 \quad \ldots\ldots(8).$$

Also $N_1 = t^4 - 2(xy)^2$, if $x^2 + y^2 = t^2$ $\ldots\ldots(9)$,

$= 2u^4 - (x \pm y)^4$, if $x^2 \pm xy + y^2 = u^2$ $\ldots\ldots(9a)$,

and N_3 is also of form N_1 $\ldots\ldots(10)$.

Hence all Fermat's numbers $F_n = E_n + 1$ where $E_n = 2^{2^n}$ are expressible in both the forms N_1, N_3 when $n > 2$; for $E_n = E_{n-1}^2 = E_{n-2}^4 = E_{n-3}^8$ when $n > 2$.

Also all numbers $M_q = 2^q - 1$, where $q = 4k + 1$ (including many of Mersenne's numbers), are of two the forms N_5 (writing $u = 1$).

Here follows a table of the least solutions (t, u) of

$$p = t_1^4 - 2u_1^2 = t_2^2 - 2u_2^4 = 2u_3^4 - t_3^2 = 2u_4^2 - t_4^4$$

for all primes $p = 8\varpi \pm 1 > 500$ (except 281). Blanks in this table merely mean that no solution in small numbers (t, u) was found, though the search has been pushed up to 100^4. Multiple solutions often exist in some one or more forms,

Ex.—$1 = 2 \cdot 1^4 - 1^2 = 2 \cdot 13^4 - 239^2$, $17 = 2 \cdot 3^2 - 1^4 = 27^2 - 3^4$,

$97 = 2 \cdot 7^2 - 1^4 = 2 \cdot 19^2 - 5^4$, $257 = 2 \cdot 13^2 - 3^4 = 2 \cdot 21^2 - 5^4$.

The table shows several (thirteen) cases of solution in three different forms; but no case of solution in each of the four forms has been found (though possibly such may exist in higher numbers t, u).

p	t_1, u_1	t_2, u_2	u_3, t_3	u_4, t_4	p	t_1, u_1	t_2, u_2	u_3, t_3	u_4, t_4
1			1, 1	1, 1	**239**	13, 119	71, 7		120, 13
7		3, 1	2, 5	2, 1	**241**				11, 1
17		7, 2		3, 1	**257**		17, 2		13, 3
23		5, 1	2, 3	18, 5	**263**			12, 203	
31	3, 5		2, 1	4, 1	**271**	19, 255	39, 5	8, 89	
41			3, 11		**281**				
47	5, 17	7, 1		8, 3	**311**				14, 3
71			4, 21	6, 1	**313**	5, 16	25, 4	7, 67	
73	3, 2				**337**	5, 12			13, 1
79	3, 1	9, 1			**353**	7, 32	19, 2	9, 113	
89	7, 34	11, 2			**359**		19, 1		
97				7, 1	**367**		23, 3		
103		115, 9	14, 277	442, 25	**383**	5, 11		6, 47	
113	5, 16	25, 4	3, 7		**401**				59, 9
127		17, 3		8, 1	**409**		21, 2	5, 29	
137		13, 2	3, 5		**431**		41, 5	4, 9	16, 3
151			4, 19	86, 11	**433**		55, 6		23, 5
167		13, 1		58, 9	**439**		21, 1		
191	11, 85		6, 49	36, 7	**449**		31, 4		15, 1
193		15, 2			**457**		93, 8		
199		19, 3		10, 1	**463**	5, 9	25, 3	4, 7	
223	7, 33	15, 1	4, 17		**479**	7, 31	511, 19		
233	5, 16	4435, 56			**487**			4, 5	38, 7

[The Proposer is indebted to Dr. T. Stuart for the two algebraic forms N_6, N_7 above, and for help in compiling the table.]

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16080. (Professor NANSON.)—R is a point on a line which passes through a fixed point O and cuts two given intersecting lines in P, Q. Having given any homogeneous relation between OP, OQ, OR, find the equation of the locus of R: for example, let OR be a mean, arithmetic, geometric, or harmonic, between OP and OQ.

Solutions (I.) *by* J. HODGKINSON; (II.) *by* C. M. ROSS.

(I.) Take O as origin and let the polar equations of the two lines be $p_1 = r\cos(\theta-a_1)$, $p_2 = r\cos(\theta-a_2)$. Let the given homogeneous relation of degree n be in the form $\Sigma A r^{n-\alpha_1-\alpha_2} r_1^{\alpha_1} r_2^{\alpha_2} = 0$ where r_1, r_2 are the values of OP, OQ for any value of θ and A is the arithmetic coefficient, *i.e.*, $\Sigma A r^{n-\alpha_1-\alpha_2} p_1^{\alpha_1} p_2^{\alpha_2} \sec^{\alpha_1}(\theta-a_1)\sec^{\alpha_2}(\theta-a_2) = 0$. Multiply by $r^n\cos^n(\theta-a_1)\cos^n(\theta-a_2)$. The equation of the locus of R then becomes $\Sigma A p_1^{\alpha_1} p_2^{\alpha_2}[r\cos(\theta-a_1)]^{n-\alpha_1}[r\cos(\theta-a_2)]^{n-\alpha_2} = 0$, a curve, in general, of degree $2n$.

E.g., $\qquad OR = \tfrac{1}{2}(OP+OQ)$,

$$2r^2\cos(\theta-a_1)\cos(\theta-a_2) = p_1 r\cos(\theta-a_2) + p_2 r\cos(\theta-a_2),$$

or in Cartesians

$$2(x\cos a_1 + y\sin a_1)(x\cos a_2 + y\sin a_2) = p_1(x\cos a_2 + y\sin a_2) + p_2(x\cos a_1 + y\sin a_1).$$

Similarly $OR^2 = OP.OQ$ gives

$$(x\cos a_1 + y\sin a_1)(x\cos a_2 + y\sin a_2) = p_1 p_2,$$

$2/OR = 1/OP + 1/OQ$ gives

$$(1/p_1)(x\cos a_1 + y\sin a_1) + (1/p_2)(x\cos a_2 + y\sin a_2) = 2.$$

(II.) Let AX, AY be the intersecting lines (included angle ω), O the fixed point, and QORP the variable line. Let

$AP = \lambda$, $\quad AQ = \mu$;

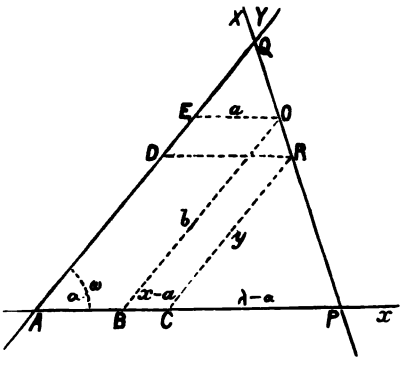
then equation of PQ is

$$x/\lambda + y/\mu = 1 \quad ...(1),$$

if O is the point (a, b)

$$a/\lambda + b/\mu = 1 \quad ...(2).$$

Again, let (x, y) be the co-ordinates of R; then, from the figure,



$$OP^2 = (\mu-b)^2 + a^2 - 2(\mu-b)a\cos\omega \quad .....(3),$$

$$OQ^2 = (\lambda-a)^2 + b^2 - 2(\lambda-a)b\cos\omega \quad .....(4),$$

$$OR^2 = (x-a)^2 + (b-y)^2 - 2(x-a)(b-y)\cos\omega \quad .....(5).$$

From (1) and (2), $\lambda = (bx-ay)/(b-y)$, $\mu = (ay-bx)/(a-x)$. Hence

$$OP^2 = a^2[(a-x)^2 + (b-y)^2 - 2(a-x)(b-y)\cos\omega]/(a-x)^2 \equiv a^2P^2/(a-x)^2,$$

$$OQ^2 = b^2[(a-x)^2 + (b-y)^2 - 2(a-x)(y-b)\cos\omega]/(b-y)^2 \equiv b^2P^2/(b-y)^2,$$

$$OR^2 = P^2.$$

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i.) Let the homogeneous relation be the arithmetic mean between OP and OQ, viz., OR; then $OP + OQ = 2OR$. Hence

$$a/(a-x) + b/(b-y) = 2,$$

neglecting factors; then $2xy = bx + ay$, a hyperbola.

(ii.) Let OR be the geometric mean between OP and OQ; then

$$OR^2 = OP \cdot OQ, \quad P^2 = abP^2/(a-x)(b-y);$$

then $xy = ay + bx$, a hyperbola.

(iii.) Let OR be the harmonic mean between OP and OQ; then

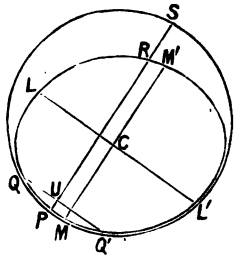
$$2/OR = 1/OP + 1/OQ;$$

therefore $2 = (a-x)/a + (b-y)/b$; therefore $x/a + y/b = 0$, a straight line. Therefore the required loci are found.

16120. (K. S. Patrachan.)—Show that the diameter of curvature at any point of an ellipse is to the normal chord at the same point in the duplicate ratio of the diameters of the ellipse respectively perpendicular and parallel to the chord.

Solutions (I.) *by* Rev. J. J. Milne, M.A.; (II.) *by* Professor Sanjána, M.A.

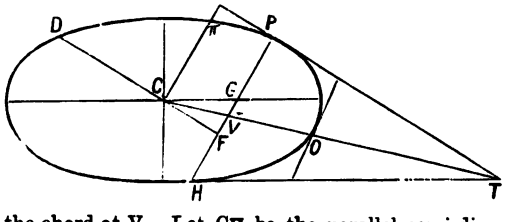
(I.) Let PR be normal chord of the ellipse, PS the diameter of curvature, QQ′ a chord parallel to the tangent at P. Then the circle QPQ′ is ultimately the circle of curvature at P. Let LL′, MM′ be the diameters parallel to the tangent and normal at P.



Then the ratio PS : PR ultimately

$$= US : UR = PU.US : PU.UR$$
$$= QU.UQ' : PU.UR = LL'^2 : MM'^2.$$

(II.) Let PGH be the normal chord at P; T its pole. Join COT,



bisecting the chord at V. Let CΠ be the parallel semi-diameter, CD the perpendicular one cutting PH at F. Then

$$PV^2/C\Pi^2 = (CO^2 - CV^2)/CO^2$$
$$= (CV.CT - CV^2)/(CV.CT) = VT/CT = VP/FP;$$

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so that $C\Pi^2 = PV.PF$. Now, by the ordinary formula, the diameter of curvature (along PG)

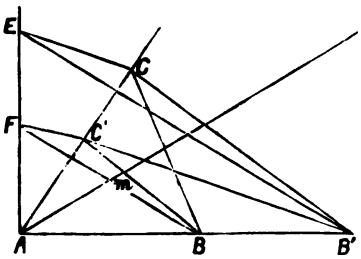
$$= 2CD^2/PF = (2CD^2.PV)/C\Pi^2 = PH(CD^2/C\Pi^2),$$

which proves the theorem.

New Proofs of some of Euclid's Theorems.

By GEORGE SCOTT, M.A.

In triangles equal in area and having an angle in common the rectangles contained by the sides about that angle are equal. —Let ACB and AC'B' be the two triangles. Since they are equal in area, BC' is parallel to B'C. On a perpendicular to AB erected at A lay off



$AE = AC$ and $AF = AC'$.

C'F will plainly be parallel to CE, and the triangles BC'F and B'CE will be equiangular; therefore FB is parallel to EB', and A*m*, which bisects BF, will also bisect B'E; in other words, the rectangles AB . AF and AB' . AE are about the same diagonal; therefore the rectangles AB' . AF and AB . AE are equal; therefore AB' . AC' = AB . AC.

Two chords intersecting at P make

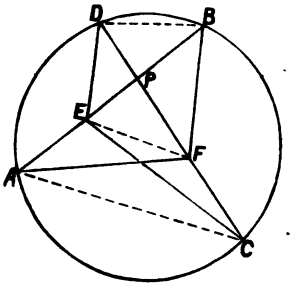
$$AP.PB = CP.PD.$$

[Proof by triangular areas.]—Make

$$PE = PD \quad \text{and} \quad PF = PB.$$

DE and BF will be parallel. Again,

$$\angle BDP = \angle PEF.$$



But BDP or BDC is equal to BAC, being in the same segment with it. Therefore EF is parallel to AC. Consequently the triangles APF and CPE are equal in area, while they have an angle in common; therefore

$$AP.PF = AP.PB = CP.PE;$$

and therefore $AP.PB = CP.PD.$

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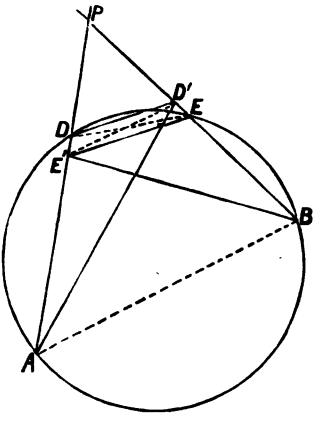
A new proof that the rectangles contained by the segments of secants drawn from a point to a circle are equal.—Let PA and PB be the two secants. Lay off

PD′ (on PB) = PD.

Make PE′ = PE. The lines DD′ and EE′ are evidently parallel, and the angles DE′D′ and DED′ are equal. But the angle DED′ is equal to the internal and opposite angle BAD of the quadrilateral ABED; therefore

∠ DE′D′ = ∠ BAD.

Therefore AB is parallel to E′D′. Hence the triangles APD′ and BPE′ are equal in area; but they have the angle APB common; therefore the rectangle



$$AP \cdot D'P = PB \cdot PE' \quad \text{or} \quad AP \cdot PD = PB \cdot PE.$$

Geometrical Note.

By H. L. Trachtenberg, B.A.

On p. 57 *supra* the following problem of mine was solved:—"If the common points of two conics lie on a circle, the circle on the line joining the points of contact of a common tangent as diameter cuts this circle orthogonally." The following is a corresponding theorem in three dimensions:—

If the common curve of two conicoids lies on a sphere, the sphere on the line joining the points of contact of a common tangent line of the two conicoids as diameter cuts this sphere orthogonally.—This theorem can be proved independently, or it can be deduced from the first theorem as follows. Let the common tangent line touch the conicoids S_1, S_2 in T_1, T_2. Let the plane through T_1, T_2 and the centre of the sphere S_0 on which lies the common curve of S_1, S_2 cut S_1, S_2 in the conics S'_1, S'_2, the sphere S_0 in the circle S'_0, and the sphere S_3 on T_1T_2 as diameter in the circle S'_3. Then, since the common points of S_1S_2 lie on the sphere S_0, the common points of $S'_1S'_2$ lie on the circle S'_0. Again, T_1T_2 is a common tangent to the conics S'_1, S'_2, T_1, T_2 being the points of contact; and S'_3 is the circle on T_1T_2 as diameter. Therefore, by the first theorem, S'_0, S'_3 cut one another orthogonally. But S'_0, S'_3 are sections of the spheres S_0, S_3 by a plane through their centres. Therefore the spheres S_0, S_3 cut one another orthogonally.

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16138. (*Communicated by* A. THOMSON.)—If $x_1, x_2, x_3, y_1, y_2, y_3$ be the roots of the cubics in x, y obtained by eliminating y, x respectively between the equations

$$(ax+hy+g)/x = (hx+by+f)/y = gx+fy+c,$$

show that $x_2x_3+y_2y_3 = x_3x_1+y_3y_1 = x_1x_2+y_1y_2 = -1.$

Solution by Professor NARANIENGAR, M.A.

Let $(ax+hy+g)/x = (hx+by+f)\,y = gx+fy+c$, each $= \lambda$, say; then

$$(a-\lambda)\,x+hy+g = 0, \quad hx+(b-\lambda)\,y+f = 0, \quad gx+fy+(c-\lambda) = 0.$$

Now, these are the equations simultaneously satisfied in order that

$$f(xy)-\lambda(x^2+y^2+1) = 0$$

may represent two straight lines. Hence the co-ordinates of the points of intersection of the pairs of straight lines formed by joining the four common points of $f(xy) = 0$ and $x^2+y^2+1 = 0$ satisfy the proposed equations. By a well known theorem, therefore, the three points of intersection are the vertices of a self-conjugate triangle. Hence, if (x_1, y_1), (x_2, y_2), (x_3, y_3) be the roots of the cubics, it follows that

$$x_1x_2+y_1y_2+1 = 0, \ \ldots.$$

An Empirical Rule for the rough calculation of the time in which a certain sum trebles itself at a given rate per cent. (compound interest), or of the rate per cent. (compound interest) at which a certain sum trebles itself in a given time.

By SARADAKANTA GANGULI, M.A.

The following rule is found by observation:—

To find the number of periods divide 111½ by the rate per cent. per period; and to find the rate per cent. per period divide 111½ by the number of periods.

The number of periods in which a certain sum trebles itself

$$\text{at } 1\% = 110{\cdot}4 = \frac{110{\cdot}4}{1}, \quad \text{at } 2\% = 55{\cdot}47 = \frac{110{\cdot}94}{2},$$

$$\text{at } 3\% = 37{\cdot}17 = \frac{111{\cdot}51}{3}, \quad \text{at } 4\% = 28 = \frac{112}{4},$$

$$\text{at } 5\% = 22{\cdot}51 = \frac{112{\cdot}55}{5}.$$

The mean of the numbers which, divided by the rates, give the corresponding periods

$$= \frac{110{\cdot}4+110{\cdot}94+111{\cdot}51+112+112{\cdot}55}{5} = \frac{557{\cdot}4}{5} = 111{\cdot}5 \text{ nearly} = 111\tfrac{1}{2}.$$

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16101. (Professor NEUBERG.)—Soient A′, B′, C′, D′ les projections des sommets A, B, C, D d'un tétraèdre sur un plan quelconque. Démontrer que les perpendiculaires abaissées des points A′, B′, C′, D′ respectivement sur les faces BCD, CDA, DAB, ABC du tétraèdre sont quatre génératrices d'un même hyperboloïde.

Solution by A. M. NESBITT, M.A.

If the four vertices have for co-ordinates

$$(l_1, l_2, l_3), \quad (m_1, m_2, m_3), \quad (n_1, n_2, n_3), \quad (r_1, r_2, r_3),$$

the four faces have for equation

$$xL_1 + yL_2 + zL_3 + L = 0, \quad xM_1 + yM_2 + zM_3 + M = 0 \ldots\ldots (l, m),$$
$$xN_1 + yN_2 + zN_3 + N = 0, \quad xR_1 + yR_2 + zR_3 + R = 0 \ldots\ldots (n, r)$$

where L_1, L_2, L_3, L, ... are the co-factors of l_1, l_2, l_3, l, ... in the determinant $\begin{vmatrix} l_1, & l_2, & l_3, & l \\ m_1, & m_2, & m_3, & m \\ n_1, & n_2, & n_3, & n \\ r_1, & r_2, & r_3, & r \end{vmatrix}$ wherein $l = m = n = r = 1$. The projections of the four points on the plane of xy are $(l_1, l_2, 0)$, ...; so that the four perpendiculars are given by $(x-l_1)/L_1 = (y-l_2)/L_2 = z/L_3$, with similar ones in m, n, and r. Now the six co-ordinates of this line are readily found to be proportional to $-L_3l_2$, L_3l_1, $L_1l_2 - L_2l_1$, L_1, L_2, L_3, and, since $L_3l_2 + \ldots + R_3r_2$ is identically zero, while

$$L_1 + \ldots + R_1 = L_1l + \ldots + R_1r,$$

and is therefore also zero, it follows that the sum of corresponding co-ordinates from the four lines vanishes. Now, if we denote the six co-ordinates by a, b, c, f, g, h with suffixes 1, 2, 3, 4 and form the equation to the hyperboloid passing through the first three of them, its coefficients (see Salmon, Art. 114) consist of a number of determinants such as $\begin{vmatrix} a_1, & b_1, & g_1 \\ a_2, & b_2, & g_2 \\ a_3, & b_3, & g_3 \end{vmatrix}$, ... But, since $\Sigma a = 0$, ..., this is the equivalent of $-\begin{vmatrix} a_1, & b_1, & g_1 \\ a_2, & b_2, & g_2 \\ a_4, & b_4, & g_4 \end{vmatrix}$, and so with the others. Hence the hyperboloid passing through the lines (l, m, n) also passes through (r).

16149. (ROBT. W. D. CHRISTIE.)—When 12345679 is multiplied by 9 we get a succession of units. Explain the reason and give the general theorem. Put also in Hensel's notation, using prime 7.

Solution by Lt.-Col. CUNNINGHAM, R.E., *and* F. E. RELTON.

If r be the radix of the scale of notation, then the number N which consists of $(r-1)$ repeated units is, of course, divisible by $(r-1)$; and

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on division, it is found that

$$N/(r-1) = [1/(r-1)]\,[(r^{r-1}-1)/(r-1)] = (111\ldots1)/(r-1)$$
$$= r^{r-3} + 2r^{r-4} + 3r^{r-5} + \ldots + mr^3 + nr^2 + or + q$$

where $q = r-1$, $p = r-2$, $o = r-3$, $n = r-4$, $m = r-5$, ... are the successive digits preceding r in the scale of r; and it is seen that the digit $p = r-2$ is omitted in the quotient of $N/(r-1)$. This gives

$$\tfrac{1}{8}[(10^9-1)/(10-1)] = \tfrac{1}{8}(111111111) = 12345679$$

(omitting 8) in scale of 10; also

$$\tfrac{1}{6}[(7^6-1)/(7-1) = \tfrac{1}{6}(111111) = 12346$$

(omitting 5) in scale of 7.

The Proposer's Solution is as follows:—

We have in the concrete

0.9	$+1 = 1$
1.9	$+2 = 11$
12.9	$+3 = 111$
123.9	$+4 = 1111$
12345678.9	$+9 =$
12345679.9	$+0 = 111111111$,

and in the abstract

$$n.10^0 + n - 1.10^1 + n - 2.10^2 + \ldots [n-(n-1)]\,10^{n-1}(10^1-1) + (n+1)\,10^0$$
$$= n.10^1 + n - 1.10^2 + n - 2.10^3 + \ldots \qquad 1.10^n + (n+1)\,10^0$$
$$-n - 1.10^1 - n - 2.10^2 - n - 3.10^3 - \ldots \qquad 0.10^n - n.10^0 + n + 1.10^0$$
$$= 1.10^0 + 1.10^1 + 1.10^2 + 1.10^3 + \ldots$$
$$= 111111111.$$

In the scale of 7 we have

$$0.6+1 = 1$$
$$1.6+2 = 11$$
$$12.6+6 = 111$$
$$12345.6+6$$
$$= 12346.6+0$$
$$= 111111 \text{ in scale of } 7;$$

and, generally,

$$1.7^4 + 2.7^3 + 3.7^2 + 4.7^1 + 5\,(7-1) + 7 - 1$$
$$= 1.7^5 + 2.7^4 + 3.7^3 + 4.7^2 + 5.7^1 \quad +7-1$$
$$-1.7^4 - 2.7^3 - 3.7^2 - 4.7^1 \quad -5$$
$$1.7^5 + 1.7^4 + 1.7^3 + 1.7^2 + 1.7^1 \quad +1;$$

and, universally,

$$r^{r-2} + r^{r-3} + r^{r-4} + \ldots + r^2 + r^1 + 1 \text{ in scale of } r$$
$$= 111111 \ldots \text{ to } (r-1) \text{ units in scale of } r.$$

15849. (Professor Sanjána, M.A.)—The diagonals of a quadrilateral ABCD meet in O. If BO = OD, and AB + BC = AD + DC, prove that the quadrilateral is a parallelogram.

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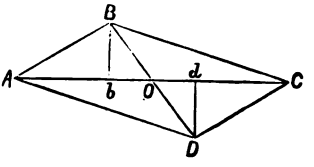
Solutions (I.) *by* J. H. TAYLOR, M.A., *and* W. E. HARRISON
(II.) *by* C. E. YOUNGMAN, M.A.

(I.) B and D are points on an ellipse of which A and C are foci. BD is either at right angles to AC or it is not. In the first case

AB = AD and CB = CD,

and the figure is symmetrical about AC. In the second case D*d* and B*b*, drawn at right angles to AC, are equal, and therefore equidistant from the foci A and C respectively. Hence AB is parallel to CD and AD is parallel to CB.



(II.) Since BO = OD, the areas ABC, ADC are equal. Hence another enunciation suggests itself, namely: Two triangles which have equal areas, equal bases, and equal sums of the other two sides are equal in all respects. For then

a, $(b+c)$, and $[(b+c)+a][(b+c)-a][a-(b-c)][a+(b-c)]$

are the same in both triangles; therefore $(b-c)$ is the same, too, and consequently b and c. Therefore either AB = AD and BC = DC or else AB = DC and BC = AD. In the first case ABCD is symmetrical about AC; in the second it is a parallelogram. Probably the Question should have been printed AO = OC instead of BO = OD: it would then follow that AB + AD = BC + DC; so that AB = AD would be excluded, except when ABCD is a rhombus.

15754. (Professor NEUBERG.)—Pour qu'une substitution

$$x = (\alpha z + \beta)/(\gamma z + \delta)$$

ramène l'équation

$$a_0x^n + \frac{n}{1}a_1x^{n-1} + \frac{n(n-1)}{1.2}a_2x^{n-2} + \ldots + a_n = 0$$

à la forme binôme $z^n - p = 0$, il faut et il suffit que les déterminants formés avec trois colonnes quelconques du tableau

$$\begin{array}{cccccc} a_0 & a_1 & a_2 & \ldots & a_{n-2} \\ a_1 & a_2 & a_3 & \ldots & a_{n-1} \\ a_2 & a_3 & a_4 & \ldots & a_n \end{array}$$

soient nuls.

Solution by C. M. Ross.

The given substitution may be written $z = (\delta x - \beta)/(\alpha - \gamma x)$. If the equation

$$a_0x^n + \frac{n}{1}a_1x^{n-1} + \frac{n(n-1)}{1.2}a_2x^{n-2} + \ldots + a_n = 0 \quad \ldots\ldots\ldots(1)$$

reduces to

$$z^n - p = 0 \quad \ldots\ldots\ldots(2),$$

(1) must be equivalent to $(\delta x - \beta)^n - p(\alpha - \gamma x)^n = 0$. Therefore, equating coefficients,

$$a_0 = \delta^n - p\alpha^n, \quad a_1 = -(\delta^{n-1}\beta - p\alpha^{n-1}\gamma),$$
$$a_2 = \delta^{n-2}\beta^2 - p\alpha^{n-2}\gamma^2 \ldots a_n = -(\beta^n - p\gamma^n).$$

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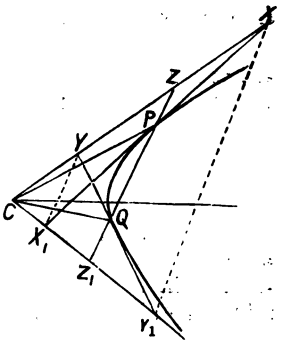
Now, if γ/α, β/δ are the roots of the equation

$$Ut^2 - Vt + W = 0 \quad \left(\begin{matrix} U\beta^2 - V\delta\beta + W\delta^2 = 0 \\ U\gamma^2 - V\gamma\alpha + W\alpha^2 = 0 \end{matrix}\right),$$

then

$$\begin{aligned} Ua_2 + Va_1 + Wa_0 &= U\delta^{n-2}\beta^2 - pU\alpha^{n-2}\gamma^2 - V\delta^{n-1}\beta + pV\alpha^{n-1}\gamma + W\delta^n - pW\alpha^n \\ &= \delta^{n-2}[U\beta^2 - V\beta\delta + W\delta^2] - p\alpha^{n-2}[U\gamma^2 - V\gamma\alpha + W\alpha^2] \\ &= 0 \qquad \ldots\ldots\ldots\

(III.) Let the chord PQ, the tangent at P, and the tangent at Q meet the asymptotes in Z, Z_1; X, X_1; Y, Y_1 respectively. Then XX_1, YY_1 are bisected at P, Q respectively, and the triangles CXX_1, CYY_1 are of constant area. Hence



$$\Delta CPX_1 = \Delta CQY.$$

But, as $ZP = QZ_1$,

therefore $\Delta CPZ_1 = \Delta CQZ$.

Therefore, by subtraction,

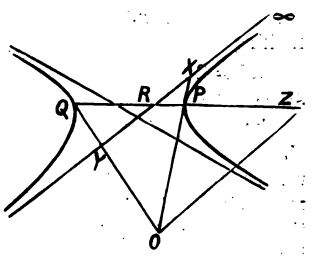
$$\Delta PZ_1X_1 = \Delta QZY;$$

and $$PZ_1 = QZ;$$

so that YX_1 is parallel to PQ. Similarly XY_1 is parallel to PQ; and, as $YQ = QY_1$ and $XP = PX_1$, it follows that PQ bisects both X_1Y_1 and XY. (F. S. Macaulay, *Geometrical Conics*, ch. iv., Exs. 15, 16.)

The Proposer demonstrates the theorem as follows:—

Draw OZ parallel to YX, meeting QP in Z. Then, O being the pole of PQ, the pole of OZ is on PQ. But the pole of YX is on YX; therefore the polar of intersection of OZ, YX passes through R, and thus OZ, OR are a pair of conjugate lines. Therefore [QRPZ] is a harmonic range. Cut the pencil O[QRPZ] by YX, and thus [YRX∞] is harmonic, or



$$YR = RX.$$

16150. (Lt.-Col. Allan Cunningham, R.E.)—Given

N^x = 549, 350, 809, 785, 675, 865, 253, 538, 956, 347, 403

to find N, x (both integers).

Solution by the Proposer.

Here N being an odd number of 33 figures ending in 3, this involves that x must be an odd number < 33, and that N must end in 3 or 7. Taking logarithms to 7 figures, and dividing by x gives

$$\log N = (1/x)(32{\cdot}7398497\ldots).$$

Writing x = 31, 29, 27, ... in succession, and finding the natural numbers (N) corresponding (which are, of course, only approximations), the only numbers (N) suitable for further trial will be those in which N is of form ($10x$ + 3 or 7) very approximately. Excluding the case of x = 3 (for which the ordinary logarithm tables do not give sufficient approx-

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imation), it is found on trial that the only value of x which gives N with close approximation to above form is $x = 11$, which gives

$$N = 947{\cdot}000\ldots;$$

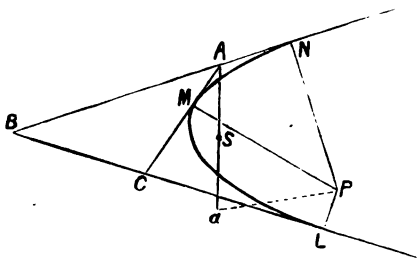
and on actual trial (by raising 947 to the 11th power) it is found that $x = 11$, $N = 947$ are the correct values.

16103. (R. F. Scott, M.A.)—Normals PL, PM, PN are drawn to a parabola, focus S. Prove that the diameter of the circle circumscribing the triangle formed by the tangents to the parabola at the points L, M, N is equal to SP.

Another Geometrical Solution of the Proposition by V. Ramaswami Aiyar, M.A.

Let ABC be the triangle formed by the tangents at L, M, N. Join AS, BS, CS, and produce them to a, b, c, making

$$AS = Sa, \ldots.$$



Since the parabola touches AM, AN at M, N, the focus S is the middle point of the symmedian chord of the circum-circle of the triangle AMN drawn through A. That is, Aa is the symmedian chord in question. Also AP is the diameter of the circle. It follows that the angle PaS is right. Similarly the angles PbS and PcS are right angles. Hence SP is a diameter of the circum-circle of the triangle abc. But this triangle is equal in all respects to ABC. Hence, &c.

The above analysis gives the Solution of (and was obtained by the Solver in connexion with) a very interesting theorem of Prof. De Longchamps given in *The Educational Times* of October, 1891, which is:—Normals being drawn to a parabola from any point P, the circle circumscribing the triangle formed by the tangents at their feet subtends a fixed angle at P.

The Proposer solves the Question as follows:—

With Mr. Nesbitt's notation, if Δ be the area of the triangle ABC,

$$2\Delta = \begin{vmatrix} amn, & a(m+n), & 1 \\ anl, & a(n+l), & 1 \\ alm, & a(l+m), & 1 \end{vmatrix} = a^2(l-m)(m-n)(n-l).$$

If R be the radius of the circle in question,

$$4R \cdot \Delta = BC \cdot CA \cdot AB \quad \text{or} \quad 2R = a\sqrt{[(1+l^2)(1+m^2)(1+n^2)]}.$$

The result follows as in Mr. Nesbitt's Solution.

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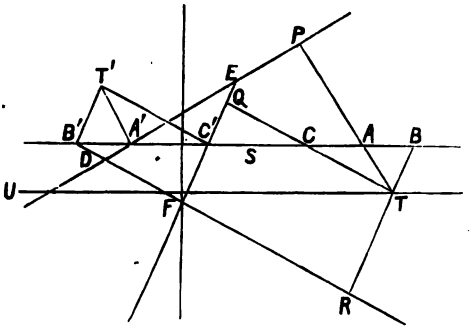
16034. (SARADAKANTA GANGULI, M.A.)—The orthocentre of the triangle formed by the tangents to a parabola at the feet of the three normals which can be drawn to it from *any* point in a given diameter of the parabola is a fixed point on the diameter.

Solutions (I.) *by* HENRY RIDDELL, M.E.; (II.) *by* F. W. REEVES, M.A., *and* C. M. ROSS.

(I.) Assume the truth of the following well known theorem:—The Simson line bisects the join of the orthocentre to the originating point



on the circum-circle. In the case of the parabola the tangent at the vertex is the Simson line of the triangle formed by any three tangents, when the focus is the point of origin. Hence the orthocentre lies always on the directrix of the parabola. Now, from T draw three normals cutting the parabola in P, Q, R and the axis in A, C, and B respectively. Draw the three tangents at P, Q, R, cutting the axis in A', C', B' and forming the triangle DEF. Now, as previously shown, the orthocentre of DEF lies on the directrix; so that, if it also lies on TU, the diameter through T, the point is fixed for all positions of T on the diameter TU. We plainly have B'A' = BA and C'A' = CA. Draw then B'T', A'T', and C'T', parallel respectively to BT, AT, and CT. These plainly meet in the point T' at the same distance from the axis as T, but on the opposite side. Now B'C' is seen to be the Simson line of DEF when T' is the point of origin. Hence the orthocentre lies on the line TU, which is the locus of the extremity of lines drawn through T' and bisected by the axis.

(II.) Let the equation of the given diameter be $y = b$. Let the three tangents be $y = -x/m_1 - am_1$, $y = -x/m_2 - am_2$, $y = -x/m_3 - am_3$. The point of intersection of the second and third is am_2m_3, $-a(m_2+m_3)$. The line through this, perpendicular to the first tangent, has for its equation $y + am_2 + am_3 = m_1(x - am_2m_3)$. This cuts the directrix where the ordinate $= a(-\Sigma m_1 - m_1m_2m_3)$ ..... (i.).

But, since the normals at the points of contact all pass through one point on $y = b$, therefore $\Sigma m_1 = 0$ and $-m_1m_2m_3 = b/a$ ..... (ii.).

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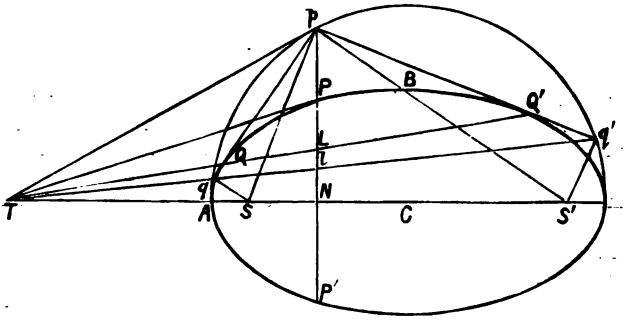
(i.) shows that the orthocentre of the triangle formed by the tangents is on the directrix $x = -a$; (ii.) combined with (i.) shows that it is on the fixed diameter $y = b$; therefore the orthocentre is the fixed point where $y = b$ cuts $x = -a$.

16144. (W. F. Beard, M.A. Suggested by Question 15976.)—P, p are corresponding points on an ellipse and its auxiliary circle. Tangents pQ, pQ' are drawn to the ellipse and meet the circle again at q, q'. Prove (1) that QQ', qq', and the tangents at P to the ellipse and at p to the circle all meet on the axis; (2) that, if the poles of QQ', qq' to the circle and of qq' to the ellipse are R, r, and R', and if pP meets QQ', qq' at L, l, then the loci of L, l, R, R', r are ellipses concentric with the original ellipse.

Solution by the Proposer.

(1) Let pP meet the axis at N, the ellipse again at P', and the circle again in p'. Let the tangent at P meet the axis at T; then pT is the



tangent at P to the circle. pP is the polar of T to the ellipse; therefore the polar of T passes through p; therefore QQ', the polar of p, passes through T. pP is the polar of T to the circle. If qq' does not pass through T, join Tq and produce it to meet pP and the circle at l_1, q_1. Then (Tql_1q_1) is harmonic; therefore p (Tql_1q_1) is harmonic. But p (Tql_1q') is harmonic; since (TQLQ') is harmonic; therefore q_1 lies on $pQ'q'$, and q_1, q' must coincide. Thus qq' passes through T.

(2) (α) Since QQ' is the polar of p to the ellipse, (pPLP') is harmonic; therefore $NP^2 = NL \cdot Np$; therefore $NL/PN = PN/pN = b/a$ (a, b being the semi-axes); therefore $NL/pN = b^2/a^2$; therefore the locus of L is an ellipse.

(β) From similar triangles Tqp, Tpq',

$$pq^2/pq'^2 = Tq^2/Tp^2 = Tq^2/(Tq \cdot Tq') = Tq/Tq',$$

and, from similar triangles Spq, S'pq', $pq/pq' = Sp/S'p$; therefore

$$Tq/Tq' = Sp^2/S'p^2 \quad \ldots\ldots (1).$$

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Now, since $(Tql q')$ is harmonic,

$$Tq/ql = Tq'/q'l = (Tq + Tq')/qq' = (Tq' + Tq)/(Tq' - Tq)$$
$$= (S'p^2 + Sp^2)/(S'p^2 - Sp^2) \text{ from (1)};$$

therefore $Tq/Tl = (S'p^2 + Sp^2)/2S'p^2$. From similar triangles SqQ, $S'pQ$,

$$qQ/pQ = Sq/S'p = (Sq \,.\, S'p)/S'p^2 = CB^2/S'p^2.$$

TQL is a transversal of the triangle pql; therefore

$$pL/Ll \,,\, Tl/Tq \,.\, qQ/pQ = 1;$$

therefore $pL/Ll = [(S'p^2 + Sp^2)/2S'p^2]\,(S'p^2/CB^2) = (2Cp^2 + 2CS^2)/2CB^2$
$= (CA^2 + CS^2)/CB^2$;

therefore pL/Ll = a constant ratio, and thus pL/pl = a constant ratio. But pN/pl = a constant ratio from (α); therefore pl/pN = a constant ratio. Thus the locus of l is an ellipse.

(γ) Since the polar of R to the circle passes through T, therefore R lies on Pp, the polar of T, and $(RpLp')$ is harmonic; therefore $NL \,.\, NR = pN^2$. But NL/pN = a constant ratio, from (α); therefore NR/pN = a constant ratio. Thus the locus of R is an ellipse. Similarly the locus of r is an ellipse.

(δ) Since the polar of R′ to the ellipse passes through T, R′ lies on pP, the polar of T, and, reasoning as in (γ), the locus of R′ is an ellipse.

9641. (T. A. Storr.)—There are 400 competitors for a prize in a lottery, each person having a number assigned to him, the numbers ranging from 1 to 400. In making the drawing three bags are used. The first, called "the units bag," contains ten balls marked 0, 1, 2, ..., 9; the second, called "the tens bag," contains the same number of balls similarly marked; and the third, called "the hundreds bag," contains five balls marked 0, 1, 2, 3, 4. A ball is drawn from the units bag and placed in the units place on a board; then one from the tens bag and placed in the tens place; and the same for the hundreds bag. The person whose number corresponds to the one drawn from the bags wins the prize. Should a greater number than 400 be drawn, the balls are replaced and the drawing made afresh. Find the expectations of those persons whose numbers range (1) from 1 to 9; (2) from 10 to 99; and (3) from 100 to 400.

Solution by Professor M. W. Crofton, F.R.S., *and* James Blaikie, M.A.

There are 500 ways in which a unit, a ten, and a hundred can be drawn from the three bags. All these ways are clearly equally probable. Therefore the chance of any number N winning in the first drawing (or any subsequent one) is $\frac{1}{500}$. There are 100 possible numbers (000 and 401—499) among the 500 which render the drawing indecisive and involve a new one; the chance of this is $\frac{100}{500} = \frac{1}{5}$. Thus the chance of N failing in the first drawing and winning in the second

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is $\frac{1}{100} \cdot \frac{1}{2}$. Likewise, that of failing in the first and second and winning in the third is $\frac{1}{100}(\frac{1}{2})^2$; and so on. Adding these probabilities, the whole chance of N winning is

$$p = \frac{1}{100}[1 + \frac{1}{2} + (\frac{1}{2})^2 + \ldots] = \frac{1}{100}/\frac{1}{2} = \frac{1}{200}.$$

Thus the chance for all numbers is $\frac{1}{200}$. We might, indeed, safely have argued that, as it is certain that one of the 400 numbers must win in the end and that the chance is the same for all, $400p = 1$.

Mr. BLAIKIE and Mr. H. M. TAYLOR, F.R.S., also consider the problem on the hypothesis that, if the number 000 is drawn, the bank is to take the prize. In this case they find the chance of a fresh drawing being required to be $\frac{99}{500}$ and the value of each competitor's expectation to be therefore

$$\left(1 + \frac{99}{500} + \frac{99^2}{500^2} + \ldots\right) \Big/ 500 = \frac{1}{401}.$$

15940. (A. M. NESBITT, M.A.)—Find the conditions that two lines—either given by a pair of points on each or a pair of planes through each—shall be polar lines with respect to a quadric given by the general equation.

Additional Solution by Professor NANSON.

Let the co-ordinates of the point x be x_1, x_2, x_3, x_4 and let the conicoid be $(xx) \equiv \Sigma a_{pq} x_p x_q = 0$. The condition that two points x, x' are conjugate is $(xx') \equiv \Sigma a_{pq} x_p x'_q = 0$. Hence the four conditions that the joins xy, $x'y'$ are polar are obviously

$$(xx') = 0, \quad (xy') = 0, \quad (yx') = 0, \quad (yy') = 0.$$

These equations may be written

$$X_1x'_1 + X_2x'_2 + X_3x'_3 + X_4x'_4 = 0, \qquad X_1y'_1 + X_2y'_2 + X_3y'_3 + X_4y'_4 = 0,$$
$$Y_1x'_1 + Y_2x'_2 + Y_3x'_3 + Y_4x'_4 = 0, \qquad Y_1y'_1 + Y_2y'_2 + Y_3y'_3 + Y_4y'_4 = 0,$$

where, in fact, X, Y are half differential coefficients of the conicoid at the points x, y. These equations show that

$$\begin{vmatrix} X_1, & X_2 \\ Y_1, & Y_2 \end{vmatrix} \begin{vmatrix} x'_1, & x'_2 \\ y'_1, & y'_2 \end{vmatrix} = \begin{vmatrix} X_3, & X_4 \\ Y_3, & Y_4 \end{vmatrix} \begin{vmatrix} x'_3, & x'_4 \\ y'_3, & y'_4 \end{vmatrix}.$$

Now (x'_1, y'_2) is one of the ray co-ordinates of the join $x'y'$ and (X_1, Y_2) is one of the axial co-ordinates of the polar line of the join xy. Thus any ray co-ordinate of $x'y'$ is proportional to the complementary axial co-ordinate of the polar line of xy. But any ray co-ordinate of a line is proportional to the complementary axial co-ordinate of the same line. Hence the above equations lead to the perfectly obvious condition that any ray co-ordinate of $x'y'$ must be proportional to the corresponding ray co-ordinate of the polar line of xy.

The analytical statement of the conditions is not without interest.

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The condition that a line with ray co-ordinate p touches the conicoid is expressed by equating to zero a certain homogeneous quadratic function ψ of the co-ordinate p, and it is readily verified that an axial co-ordinate q of the polar line of a line with ray co-ordinate p is given by $q = \frac{1}{2}\partial\psi/\partial p$. It follows then that the required conditions may be expressed by saying that any co-ordinate of the one line must be proportional to the differential coefficient of ψ with respect to the complementary co-ordinate of the other line.

The PROPOSER discusses the Question as follows:—

Let $P_1(x_1y_1z_1w_1)$ and $P_2(x_2y_2z_2w_2)$ determine one line, P_3 and P_4 the other. Now the polar plane of $(\xi\eta\zeta\Omega)$ is $\xi\, dF/dx + \ldots = 0$; and, if ξ lies on P_1P_2, this passes through P_3 and P_4; *i.e.*,

$$(x_1 + \theta x_2)\, dF/dx_3 + \ldots = 0 \quad \text{and} \quad (x_1 + \theta x_2)\, dF/dx_4 + \ldots = 0$$

for all values of θ. Hence the conditions are

$$x_1\, dF/dx_3 + \ldots = 0, \qquad x_2\, dF/dx_3 + \ldots = 0,$$
$$x_1\, dF/dx_4 + \ldots = 0, \qquad x_2\, dF/dx_4 + \ldots = 0.$$

Since these may also be written $x_3\, dF/dx_1 + \ldots = 0, \ldots$, the relation is, of course, a reciprocal one.

It may be remarked that this will furnish a simple solution of Professor Nanson's Question 15762. For, if the above relations be briefly written as $(1, 3) = 0$, $(2, 3) = 0$, $(1, 4) = 0$, $(2, 4) = 0$, then we get from the fact that P_1P_3 and P_2P_4 are polar lines the relations

$$(1, 2) = 0, \quad (1, 4) = 0, \quad (3, 2) = 0, \quad (3, 4) = 0.$$

Hence we have the four

$$(1, 2) = 0, \quad (1, 3) = 0, \quad (2, 4) = 0, \quad (3, 4) = 0;$$

but, seeing that the last two may be written in the equivalent forms $(4, 2)$ and $(4, 3)$, these exactly express the condition that P_1P_4 and P_2P_3 are also polar lines.

There is no special need to discuss the case when the two original lines are given by pairs of planes; since clearly the condition that the lines L_1L_2 and L_3L_4 shall be conjugate are

$$l_1\, d\Sigma/dl_3 + \ldots = 0, \qquad l_2\, d\Sigma/dl_3 + \ldots = 0,$$
$$l_1\, d\Sigma/dl_4 + \ldots = 0, \qquad l_2\, d\Sigma/dl_4 + \ldots = 0;$$

$l_1x + \ldots + r_1w = 0$ being the equation to L_1, ..., and Σ being

$$Al^2 + \ldots + \ldots + 2Fmn + \ldots + \ldots + 2Prl + \ldots + \ldots = 0.$$

The proof works out precisely the same for tangential ("plane") co-ordinates as for tetrahedral ("point") co-ordinates.

10589. (Professor EMMERICH.)—Construct an isosceles triangle, knowing any three of the following points:—A (vertex), G (centre of gravity), K (Lemoine point), Ω, Ω' (Brocard points).

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Solution by Professor SANJÁNA, M.A.

(1) When A, G, K are given then (AB $\neq$ AC) draw AD perpendicular to GK and make GB (in DG produced) = 2GD: B is the extremity of both equal sides. But, if AB = AC, then A, G, K are collinear. In AG produced take GD = $\frac{1}{4}$AG, and draw a line through D at right angles to AD. Also draw the circle which is the locus of points at which AG and KD subtend equal angles, and let this cut the perpendicular at D in B: B is the extremity of one of the equal sides.

(2) If Ω, Ω', k be given, then the Brocard angle is known, whence the circum-centre can be found. The Brocard ellipse and circum-circle being drawn, the triangle may be any one of the circumscribed series: of these take the maximum and minimum, which are the required isosceles triangles. (See Milne, *Companion*.) Other cases may be similarly treated.

16125. (Professor NANSON.)—If the perpendiculars from the vertices of one tetrahedron upon the faces of a second are concurrent, so are the perpendiculars from the vertices of the second upon the faces of the first.

Solution by A. M. NESBITT, M.A.

Let λ, μ, ν, ρ be the vertices of the first tetrahedron, the co-ordinates of λ being $(l_1\lambda, l_2\lambda, l_3\lambda)$, with similar values for those of μ, ν, ρ. Let the corresponding faces of the second tetrahedron be $l = 0$, $m = 0$, $n = 0$, $r = 0$, where $l \equiv l_1x + l_2y + l_3z + 1$, ... Let the two determinants

$$\begin{vmatrix} l_1, & l_2, & l_3, & l \\ m_1, & m_2, & m_3, & m \\ n_1, & n_2, & n_3, & n \\ r_1, & r_2, & r_3, & r \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \lambda l_1, & \lambda l_2, & \lambda l_3, & l \\ \mu m_1, & \mu m_2, & \mu m_3, & m \\ \nu n_1, & \nu n_2, & \nu n_3, & n \\ \rho r_1, & \rho r_2, & \rho r_3, & r \end{vmatrix}$$

be respectively denoted by Δ and ∇; in both of which

$$l = m = n = r = 1.$$

Denote the co-factors of the elements in the several rows of Δ by (L_1, L_2, L_3, L), $(M_1, \ldots)$, $(N_1, \ldots)$, $(R_1, \ldots)$; those of the elements of ∇ by $(L'_1, \ldots, L')$, $(M'_1, \ldots, M')$, $(N'_1, \ldots, N')$, $(R'_1, \ldots, R')$.

Now, the co-ordinates of the point (m, n, r) will be L_1/L, L_2/L, L_3/L; and the equation to the plane $(\mu\nu\rho)$ may be got by writing $x, y, z, 1$ for the first row in ∇; *i.e.*, it is $xL'_1 + yL'_2 + zL'_3 + L' = 0$; therefore the equations of the line through mnr perpendicular to the plane $\mu\nu\rho$ are $(Lx - L_1)/L'_1 = (Ly - L_2)/L'_2 = (Lz - L_3)/L'_3$.

So, too, the line through νrl perpendicular to $\nu\rho\lambda$ has for equations

$$(Mx - M_1)/M'_1 = (My - M_2)/M'_2 = (Mz - M_3)/M'_3;$$

and the condition that these two lines intersect is

$$\begin{vmatrix} L_1M - LM_1, & L_2M - LM_2, & L_3M - LM_3 \\ L'_1, & L'_2, & L'_3 \\ M'_1, & M'_2, & M'_3 \end{vmatrix} = 0.$$

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Now $$\begin{vmatrix} L'_2, & L'_3 \\ M'_2, & M'_3 \end{vmatrix} = \nabla^3 \begin{vmatrix} pn_1, & 1 \\ pr_1, & 1 \end{vmatrix}$$

and $$L_1M - LM_1 = \begin{vmatrix} L_1, & L \\ M_1, & M \end{vmatrix} = \Delta^3 \begin{vmatrix} n_2, & n_3 \\ r_2, & r_3 \end{vmatrix}$$

by a property connecting adjugate determinants.

Hence the above condition is satisfied identically, and similarly every one of the four lines we are considering intersects every other. But the four clearly do not lie in the same plane; therefore they are concurrent.

16033. (J. HODGKINSON. *Suggested by* Question 15976.) — Two conics are drawn, one fixed, one variable, each having double contact with a fixed conic. Show that (1) the intersection of one pair of common chords of the two conics always lies on a fixed straight line, (2) one diagonal of the quadrilateral formed by the four common tangents of the two conics always passes through a fixed point.

Solution by the PROPOSER.

The theorems are reciprocal. The first is here proved geometrically, the second analytically.

(1) Two conics having double contact have an infinite number of common self-conjugate triangles, the common chord of contact being always one side, while any point on it may be a vertex. Take the intersection of the common chords of contact. Then, with this point as vertex, a common self-conjugate triangle can be drawn with respect to each of the two pairs of conics having double contact. These two triangles must plainly have the side opposite to this vertex in common, since this side is in each case the polar of the vertex for the fixed conic. We have thus found a point such that it has the same polar for the other two conics. It is therefore the intersection of a pair of their common chords, and, being taken at the intersection of a fixed, straight line and a variable, must always lie on the fixed straight line.

(2) Let the tangential equation of the fixed conic be $\Sigma = 0$ and $L = 0$ the tangential equation of the intersection of the common tangents with the fixed conic having double contact. Then the equation of this conic is $\Sigma + \lambda L^2 = 0$ where λ is a constant. The equation of the second conic is $\Sigma + \mu M^2 = 0$ where $M = 0$ is the equation of any point. Then the equation $\lambda L^2 = \mu M^2$ gives a point pair touching the four common tangents of these two conics, and the line joining them is found from the equations $\sqrt{\lambda} L = \pm \sqrt{\mu} M$, which clearly always passes through the point $L = 0$.

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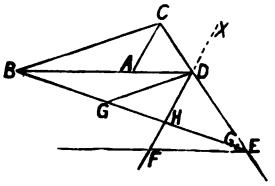
11827. (J. E. HENDRICKS.) — To construct a triangle ABC (BC > AB > CA), the angle A being known, and having given

$$AB+BC=m,\quad AB+AC=n,\quad \text{or}\quad BC-AB=p,\quad AB-AC=q.$$

Solutions (I.) *by* W. F. BEARD, M.A.; (II.) *by* HENRY RIDDELL, M.E.

(I.) To construct a triangle ABC, given the angle A, $a+c=m$, $b+c=n$ $(a>c>b)$.—Draw BD = n. Make

$$\angle BDX = A.$$



Produce XD to F, making

$$DF = m-n.$$

Draw FE parallel to BD to meet the bisector of the angle BDX at E. Join BE. With centre D, radius DF, describe a circle cutting BE at G, G_1. Draw BC parallel to DG to meet ED at C. Draw CA parallel to DF. Then ABC is the triangle required.

By similar triangles

$$BC/CE = DG/DE = DF/DE$$
$$= CA/CD \text{ (by similar triangles)}$$
$$= (BC-CA)/DE;$$

therefore $$BC-CA = DF = m-n \quad \ldots\ldots(1).$$

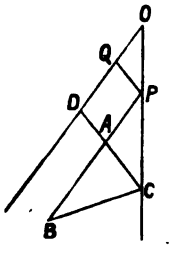
Also $\angle ACD = \angle CDX$ (by parallels) $= \angle CDA$; therefore AC = AD; therefore $AB+AC = BD = n$; therefore, from (1), $AB+BC = m$, and $\angle BAC = \angle BDX$ (by parallels) = A. Thus ABC is the triangle required.

A similar construction will plainly hold when A, $a-c$, $c-b$ are given.

We can prove that only one triangle can be found to satisfy the conditions. Let DF meet AE at H. Then DG and DG_1 are each greater than DH, since each is equal to DF. Thus DG, DG_1 must lie on opposite sides of DHF. Thus, if BC_1 is drawn parallel to DG_1 and C_1A_1 drawn parallel to DF, A_1 would lie in DB produced, and $\angle C_1A_1B_1$ must then = 180°−A.

(II.) Since $AB+BC = m$ and $AB+AC = n$, we have $BC-AC = m-n$. From any point O draw a line of indefinite length OD, and cut off

$$OQ = m-n.$$



Draw also the line OC, making the angle $DOC = \frac{1}{2}A$. Inflect QP on this line = QO. Draw PB parallel to OD and = n. With OB as base, draw the locus of vertices of triangles having their sides in the ratio QP/PO. Let this locus cut OC in C. Join BC and draw CA parallel to QP; then is ABC the triangle required. For BC/CO = QP/PO = DC/CO. Therefore BC = DC. And DA = QP = OQ = $m-n$. Also AC = AP and BA + AC = BP = n. Therefore the triangle ABC answers the conditions of the problem.

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15719. (J. J. BARNIVILLE, B.A., I.C.S.)—Having $u_n + 2u_{n+1} = u_{n+2}$, prove $$\frac{2.11.29}{1.3.7.17} + \frac{5.27.69}{1.7.17.41} + \frac{12.65.167}{3.17.41.99} + \dots = \frac{67}{14}.$$

Solution by the PROPOSER.

Since x^2+2x-1 is a factor of x^4-4x^2+4x-1, we have

$$u_n - u_{n-4} = 4u_{n-1} - 4u_{n-2}.$$

Let $$S_1 = \frac{2}{1.17} + \frac{5}{1.41} + \frac{12}{3.99} + \dots = \frac{13}{42}$$

and $$S_2 = \frac{2}{3.7} + \frac{5}{7.17} + \frac{12}{17.41} + \dots = \frac{1}{6};$$

then the given series $= 16S_1 - S_2 = \frac{67}{14}$.

Similarly $$\frac{7.19.49}{1.5.12.29} + \frac{17.46.118}{2.12.29.70} + \frac{41.111.285}{5.29.70.169} + \dots = \frac{104}{15}.$$

16160. (I. ARNOLD.)—Three points A, B, C are in one right line, and a circle is drawn with A as centre. Find the locus of the intersection of the right line passing through B and one extremity of a diameter of the circle and the right line passing through C and the other extremity of the same diameter.

Solutions (I.) *by* V. DANIEL, B.Sc., *and* W. J. DOBBS, M.A.; (II.) *by* JAMES BLAIKIE, M.A., *and others.*

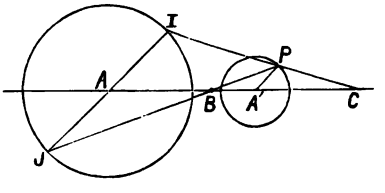
(I.) Let O be the intersection of the lines, and let OK be drawn parallel to the variable diameter to meet ABC in K. Since the extremities of the diameter, its mid-point A, and the point at infinity on it form a harmonic range, their joins to O form a harmonic pencil. Therefore (ABKC) is a harmonic range. Therefore K is a fixed point. Also, if ρ = radius of fixed circle, by similar triangles,

$$OK/\rho = BK/AB = (b-c)/(b+c),$$

where $AB = c$, $AC = b$. Therefore locus of O is a circle with centre at K (the harmonic conjugate of A with respect to BC), and

$$\text{radius} = [(b-c)/(b+c)]\,\rho.$$

(II.) Let IJ be a diameter through A, and let JB, IC meet in P. Draw PA′ parallel to IJ, meeting ABC in A′. The circle described with A′ as centre and A′P as radius is the required locus.



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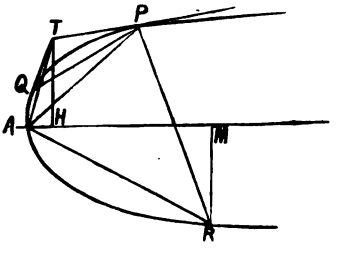
For B and C are clearly the centres of similitude of the circles A and A', since A'P is parallel to the radii AI, AJ, and, if another diameter I'AJ' and another radius A'P' are drawn, I'P' will clearly go through C and J'P' will go through B.

16105. (A. M. NESBITT, M.A.)—A chord of a parabola is such that the normals at its extremities meet on the curve. Show, by pure geometry, that the pole of the chord lies on a certain straight line.

Solutions (I.) *by* JAMES BLAIKIE, M.A.; (II.) *by* M. T. NARANIENGAR, M.A.; (III.) *by* HENRY RIDDELL, M.E.

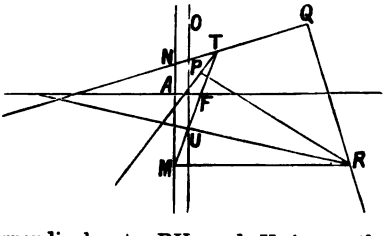
(I.) In Milne and Davis, *Conics*, Art. 78, it is shown that the chord in question passes through a fixed point on the axis — namely, the image of the vertex in the directrix. It follows that its pole lies on the chord of contact of the tangents drawn from this point, which is the ordinate through a point on the axis found by producing AS its own length.

(II.) We know that, if PQ be the chord, and PR, QR the normals at P and Q, the sum of the ordinates of P and Q is equal to that of R. So that $2TH = RM$ (T being the pole of PQ). Again, because PQRA is cyclic, the angle TAR is a right angle, and therefore $TH/AH = AM/RM = RM/4a$; therefore $AH = 2a = \text{const.}$, or the locus of T is a straight line perpendicular to the axis.



(III.) This theorem is found as an exercise in several text-books: for example, in Taylor's *Geometrical Conics*. It is best proved by anharmonics; but the following proof, depending on the theorem by Saradakanta Ganguli, M.A. (Question 16084), may be new:—

TP, TQ are two tangents; PR, QR the corresponding normals meeting in R on the parabola. Draw the tangent and diameter at R. The orthocentre of the triangle formed by the tangents lies at M, the intersection of the diameter at R and directrix (Question 16084). Therefore TM is perpendicular to RU, and U is on the



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tangent at the vertex, since AF = the subnormal belonging to R, and therefore F is the focus of the curve.

The points P, T, Q, R, U are concyclic. Also a circle through T and the mid-points of PT and PQ passes through the focus, and has its diameter half that of the previous circle, and, as it touches the same at T, every chord through T is bisected by the circles. Hence TF = FU, and the locus of T is a line parallel to the tangent at the vertex, and at a distance from it equal to AF.

The following Solution is due to the PROPOSER:—

Let R be a point on the parabola, and RQ, RQ′ the normals drawn from it. Then the four points R, A, Q, Q′ are concyclic, and, consequently, if the tangents at Q and Q′ meet at P, PAR is a right angle. Let Q′Q meet RA in X and the axis in T; also let XO, PM, and RN be drawn perpendicular to the axis, and let RZ, at right angles to RA, meet the axis in Z. Now, since XQ.XQ′ = XA.XR, the chords are

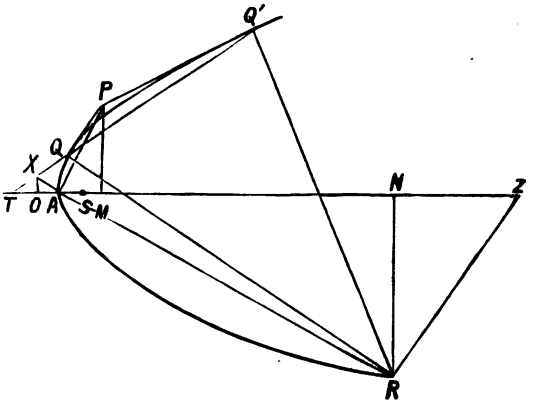

FIG. 1.

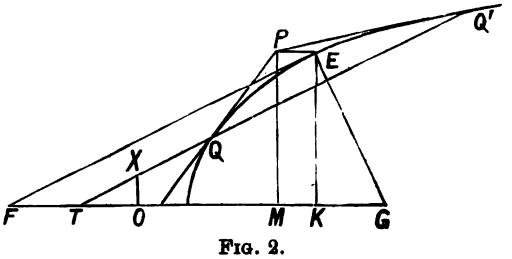

FIG. 2.

equally inclined to the axis; whence TO = OA. Also

$$RN^2 = 4AS.AN = AN.NZ;$$

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so that NZ = 4AS; while we have also, from elementary principles,*

$$XO/OT = 2AS/PM.$$

But $$XO/OT = XO/OA = RN/NA = NZ/RN;$$

therefore RN = 2PM; so that AM = ½NZ = 2AS. Hence the locus of P is the straight line $x = 2a$.

Arithmetical Note.

By SHISHIR KUMAR MAITRA, B.A.

To show that, if, and only if, n be a prime number,

$$(n-1)(2n-1) \ldots [(n-2)n-1]$$

is divisible by $(n-1)!$.

We propose first to show that, if b be any number less than n, then the series $(n-1)(2n-1)(3n-1) \ldots [(b-1)n-1]$ contains one, and only one, term divisible by b.

Since n is a prime number, therefore when

$$n-1,\ 2n-1,\ \ldots,\ (b-1)n-1$$

are divided by b the remainders are all different.

Hence as there are $b-1$ terms, therefore there are $b-1$ different remainders.

Now when any number is divided by b the possible remainders are b in number, namely, 0, 1, 2, ..., $(b-1)$.

But, in our case, $b-1$ is not a possible remainder; for, if $(rn-1)$—the general term of the series $(n-1)(2n-1) \ldots [(b-1)n-1]$—leave $(b-1)$ as remainder when divided by b, then rn would be divisible by b, which is impossible, since n is a prime number and r is less than b.

Hence, in our case, the remainders are 0, 1, 2, ..., $(b-2)$, occurring in some order or other.

Therefore, one, and only one, term of the series

$$(n-1)(2n-1) \ldots [(b-1)n-1]$$

is divisible by b.

We propose next to show that the highest power of any prime number p contained in $(n-1)(2n-1) \ldots [(n-2)n-1]$ is at least equal to the highest power of that prime number contained in $(n-1)!$.

Since, by supposition, p occurs in $(n-1)!$, therefore p is less than n.

Hence, from what we have already proved, it follows that one, and only one, term of the series $(n-1)(2n-1) \ldots [(p-1)n-1]$ is divisible by p.

Let $rn-1$ be that term of the series

$$(n-1)(2n-1) \ldots [(p-1)n-1]$$

which is divisible by p. Then the number of terms divisible by p

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* This is really an elementary principle. For, if the diameter through P (Fig. 2) meet the parabola in E, and if EF, EG be respectively the tangent and normal at E, and EK the ordinate of E, we have XO/OT = EK/KF = KG/EK = 2AS/PM, since PE is parallel to the axis.

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which occur in the series $(n-1)(2n-1)\ldots[(n-2)n-1]$ may be thus calculated :—

In the first r terms of the series $(n-1)(2n-1)\ldots[(n-2)n-1]$ there is one, and only one, term divisible by p, namely, $rn-1$. In the remaining $n-2-r$ terms there are $\mathrm{I}[(n-2-r)/p]$ terms divisible by p, since in every p consecutive terms there is one, and only one, term divisible by p. Hence, the total number of terms divisible by p which occur in the series $(n-1)(2n-1)\ldots[(n-2)n-1]$ is

$$1+\mathrm{I}[(n-2-r)/p].$$

Now, as $rn-1$ is a term of the series

$$(n-1)(2n-1)\ldots[(p-1)n-1],$$

r is never greater than $p-1$. Consequently $\mathrm{I}[(n-2-r)/p)]$ is never less than $\mathrm{I}\{[n-2-(p-1)]/p\}$, *i.e.*, never less than $\mathrm{I}[(n-1-p)/p]$, *i.e.*, never less than $\mathrm{I}[(n-1)/p]-1$.

Therefore $1+\mathrm{I}[(n-2-r)/p]$ is never less than $\mathrm{I}[(n-1)/p]$. That is to say, the number of terms divisible by p which occur in the series $(n-1)(2n-1)\ldots[(n-2)n-1]$ is never less than $\mathrm{I}[(n-1)/p]$.

Similarly, the number of terms divisible by p^2 which occur in the same series is never less than $\mathrm{I}[(n-1)/p^2]$; and so on.

Hence, the highest power of p contained in

$$(n-1)(2n-1)\ldots[(n-2)n-1]$$

is never less than $\mathrm{I}[(n-1)/p]+\mathrm{I}[(n-1)/p^2]+\ldots+\mathrm{I}[(n-1)/p^s]$ where s is the highest power of p consistent with the condition that p^s is not greater than $n-1$.

But $\mathrm{I}[(n-1)/p]+\mathrm{I}[(n-1)/p^2]+\ldots+\mathrm{I}[(n-1)/p^s]$ is the highest power of p contained in $(n-1)!$. Therefore, the highest power of p contained in $(n-1)(2n-1)\ldots[(n-2)n-1]$ is at least equal to the highest power of p contained in $(n-1)!$. Therefore $(n-1)(2n-1)\ldots[(n-2)n-1]$ is divisible by $(n-1)!$.

We propose to show next that the theorem is true only when n is a prime number. For, suppose n is not a prime number and has A for one of its prime factors. Then, as A is a prime number, and, as none of the terms $n-1$, $2n-1$, ..., $(n-2)n-1$ is divisible by A, the product $(n-1)(2n-1)\ldots[(n-2)n-1]$ is not divisible by A. Hence, clearly, the product is not divisible by $(n-1)!$.

Note 1.—It may be shown in exactly the same way as before that $(n+1)(2n+1)\ldots[(n-2)n+1]$ is divisible by $(n-1)!$ when, and only when, n is a prime number.

Note 2.—If n be a prime number, and if q be the quotient obtained by dividing $(n-1)(2n-1)(3n-1)\ldots[(n-2)n-1]$ by $(n-1)!$, then $q-1$ is divisible by n.

By Wilson's theorem, $(n-1)!$ is of the form $\mathrm{M}(n)-1$. Also,

$$(n-1)(2n-1)\ldots[(n-2)n-1]$$

is of the form $\mathrm{M}(n)-1$, because $n-2$ is odd when n is a prime number (except when $n=2$). Therefore q is of the form $\mathrm{M}(n)+1$. That is, $q-1$ is divisible by n.

Similarly, it may be shown that, if n be a prime number, and if q' be the quotient obtained by dividing $(n+1)(2n+1)\ldots[(n-2)n+1$ by $(n-1)!$, then $q'+1$ is divisible by n.

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10761. (Professor CATALAN.)—Si a, b, c désignent les côtés d'un triangle rectiligne, démontrer que $a^2b(a-b)+b^2c(b\ \ c)+c^2a(c-a)\equiv 0$.

[$\equiv$ is doubtless a misprint and should be $\geqslant$.—ED.]

Remark by Professor SANJÁNA, M.A., C. M. ROSS, *and others.*

It would seem there is some defect in the enunciation. The given expression can be shown to be in general greater than zero. For let in general $a>b>c$, so that $a-b=x$, $b-c=y$, and x and y are positive. Expressing in terms of c, x, y, we get the expression

$$= (c+x+y)^2(c+y)x+(c+y)^2cy\quad c^2(c+x+y)(x+y)$$
$$= (c+x+y)x(2cy+y^2+xc+xy)+cy(2cy+y^2-cx-cy)$$
$$= c^2xy+cx^2y\cdot cxy^2+x(c+x+y)(y+x)(y+c)+c^2y^2+cy^3\cdot c^2xy$$
$$= cy(x^2+xy+y^2+cy)+x(c+x+y)(y+x)(y+c);$$

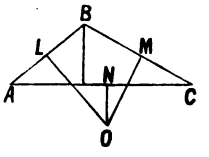
every symbol is positive and all the signs are *plus*.

The left-hand obviously vanishes when $a=b=c$.

16165. (Professor R. W. GENESE, M.A.)—A force acts at each vertex of a rigid polygon at right angles and proportional to the join of the adjacent vertices. Then (1) the system balances; (2) if each force retain its original direction, but the polygon be turned through an angle θ, the system is equivalent to a couple whose moment varies as $\sin\theta$.

Solution by M. T. NARANIENGAR, M.A.

(1) Let A, B, C be three consecutive vertices: then the force along the perpendicular on AC proportional to AC is equivalent to forces at B perpendicular to AB and BC, and proportional to AB, AC respectively. Similarly with each force. Hence the given system reduces to forces at the extremities of each side perpendicular to the side and proportional to it. These are again equivalent to forces at the mid-points of the sides perpendicular and proportional to the sides. Now, if L, M, N be the mid-points of AB, BC, AC, forces at L and M perpendicular to AB, BC and proportional to them have for resultant a force at N along NO proportional to AC (by triangle of forces). Thus, forces at the mid-points of AB, BC, CD will be equivalent to a similar force at the mid-point of AD, and so on. Hence the resultant of all but one will be equal and opposite to the last, and therefore the system balances.



(2) Next, a force λ AB at the mid-point of AB, when turned through the angle θ may be resolved into (λ AB cos θ) and (λ AB sin θ) perpendicular to AB and along AB respectively. By the first part all such components as λ AB cos θ balance, and the latter are equivalent to a couple whose moment is $\lambda\Delta\sin\theta$, where Δ is the area of the polygon. Hence, &c.

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16155. (Professor NANSON. Suggested by Question 16014.)—Find the locus of the foci of a conic of given eccentricity described about a given triangle.

Solution by the Hon. G. R. DICK, M.A.

Denoting the given eccentricity by e, let ρ, σ, τ be the distances of a focus from the three points A, B, C; then the perpendiculars from A, B, C to the corresponding directrix are ρ/e, σ/e, and τ/e respectively. Now, if α, β, γ denote the perpendiculars from the vertices of a triangle ABC on any straight line whatever, we have the well known relation

$$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 - 2bc\cos A\beta\gamma - 2ca\cos B\gamma\alpha - 2ab\cos C\alpha\beta = 4\Delta^2,$$

a, b, c, Δ denoting the sides and area of ABC.

Substituting ρ/e, ..., for α, β, γ, we get

$$a^2\rho^2 + b^2\sigma^2 + c^2\tau^2 - 2bc\cos A\sigma\tau - 2ca\cos B\tau\rho - 2ab\cos C\rho\sigma = 4e^2\Delta^2.$$

This, then, is the equation of the locus. If x_1, y_1, ... are the coordinates of the vertices, $\rho^2 = (x-x_1)^2 + (y-y_1)^2$, ..., replacing ρ^2, ... by $(x-x_1)^2 + (y-y_1)^2$ and rationalizing, the locus is found to be a sextic. The terms of the seventh and eighth degrees are easily shown to vanish, and when $e = 1$ the sixth powers disappear. This is, of course, in accordance with the results obtained by Professor Hudson in *The Educational Times* for December, 1906, which show that in the case of a parabola the locus is an *unicursal quintic*. It has three asymptotes parallel to the sides of the triangle, cutting each side at a distance of one-fourth from the nearest angular point.

The equation

$$a^2\rho^2 + b^2\sigma^2 + c^2\tau^2 - 2bc\cos A\sigma\tau - 2ca\cos B\tau\rho - 2ab\cos C\rho\sigma = 4\Delta^2$$

(in the case of a parabola) is convenient for obtaining the Cartesian (or polar) equation in special cases, such as when the given points form an equilateral triangle, for instance.

The PROPOSER'S Solution is the following:—

From the focus and directrix definition of a conic and from the well known relation connecting the distances of a straight line from the vertices of a triangle, it follows at once that the equation of the required locus is

$$a^2(x-y)(x-z) + b^2(y-z)(y-x) + c^2(z-x)(z-y) = 4e^2\Delta^2,$$

where x, y, z are the distances of a focus from the vertices, e is the eccentricity, a, b, c are the sides, and Δ is the area of the triangle.

9186. (F. MORLEY, B.A.)—Solve the equations

$$(y-z)(y^2-z^2) = a, \quad (z-x)(z^2-x^2) = b, \quad (x-y)(x^2-y^2) = c.$$

[Solutions too numerous to allow of all being published.—ED.]

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Purely Algebraical Solutions (I.) *by* S. T. SHOVELTON, M.A.; (II.) *by* D. BIDDLE.

(I.) Write $2x = u+v$, $2y = u-v$, $2z = u+v+2\theta$.

Then $uv^2 = c$, $\theta^2(u+v+\theta) = b$, $(u+\theta)(v+\theta)^2 = a$;

therefore $\theta^2 v + \theta(2uv+v^2) = a-b-c$;

i.e., $\theta v(\theta+v+2u) = a-b-c$ or $\theta v(b/\theta^2 + c/v^2) = a-b-c$;

therefore v/θ is a root of the equation

$$b\xi^2 - (a-b-c)\xi + c = 0;$$

whence the values of u, v, θ may readily be found.

(II.) Let $y = x+\lambda$, $z = x+\lambda+\mu$; then

$$\mu^2(2x+2\lambda+\mu) = a, \quad (\lambda+\mu)^2(2x+\lambda+\mu) = b, \quad \lambda^2(2x+\lambda) = c;$$

whence $(3\lambda+\mu+4x)\lambda\mu = b-c-a.$

But $3\lambda+\mu+4x = c/\lambda^2 + a/\mu^2;$

therefore $c\mu/\lambda + a\lambda/\mu = b-c-a;$

and $\mu = 1/2c\,[(a^2+b^2+c^2-2ab-2ac-2bc)^{\frac{1}{2}} - (a-b+c)]\,\lambda = f\lambda$, say,

where f is found. Now we have

$$x = (c-\lambda^3)/(2\lambda^2) = (a-\mu^3-2\lambda\mu^2)/(2\mu^2),$$

whence $a\lambda^2 - c\mu^2 = \lambda^3\mu^2 + \lambda^2\mu^3.$

Substituting $f\lambda$ for μ, we obtain

$$a-cf^2 = (f^2+f^3)\lambda^3;$$

therefore $\lambda = [(a-cf^2)/(f^2+f^3)]^{\frac{1}{3}}$, $\mu = f[(a-cf^2)/(f^2+f^3)]^{\frac{1}{3}}$.

Also $x = (c-\lambda^3)/(2\lambda^2) = \{c-[(a-cf^2)/(f^2+f^3)]\}/\{2[(a-cf^2)/(f^2+f^3)]^{\frac{1}{3}}\}$,

$y = x+[(a-cf^2)/(f^2+f^3)]^{\frac{1}{3}}$, $z = x+(1+f)[(a-cf^2)/(f^2+f^3)]^{\frac{1}{3}}$.

Solution by aid of Analytical Geometry by T. STUART, M.A., D.Sc.

We have evidently

$$\Sigma[a/(y-z)] = 0, \quad \Sigma[a/(y^2-z^2)] = 0;$$

that is $\Sigma ax^2 + \Sigma(a-b-c)yz = 0$ .....(1),

and $\Sigma ax^4 + \Sigma(a-b-c)y^2z^2 = 0$ .....(2).

Since the discriminants of these equations vanish, they each break up into two factors and give us

$$2ax + y(c-a-b \pm a_0) + z(b-c-a \mp a_0) = 0 \text{ ..... (3)},$$

$$2ax^2 + y^2(c-a-b \pm a_0) + z(b-c-a \mp a_0) = 0 \text{ ..... (4)},$$

where $a_0^2 \equiv \Sigma a^2 - 2\Sigma bc$.

If we take the upper sign in both, we see that one of the lines represented by (3) touches at the point (1, 1, 1) one of the conics represented by (4), and the same is true if the lower signs in both are taken. If unlike signs are taken, the intersections are distinct (unless a_0 vanishes). Hence in addition to the solution (1, 1, 1) there are two solutions of the equations (3), (4), and consequently six solutions of the original equations. The explicit values of x, y, z in terms of a, b, c are at once deducible.

16167. (W. J. GREENSTREET, M.A.)—Solve

$$(y^3+2xy^2)y' + (x+y)(xy'-y) = 2y^3.$$

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Solution by Dr. T. STUART.

This equation is only a particular case of the more general differential equation of the first order,

$$Ly' + M(xy' - y) = N \quad \ldots\ldots\ldots\ldots\ldots\ldots (\alpha),$$

where L, M, N are homogeneous functions of x and y. This equation can easily be solved when L and N are of the same degree (as in the Question proposed) by means of the substitution $y = vx$. Writing

$$L = x^n\theta(y/x) = x^n\theta(v), \quad M = \ldots = x^m\phi(v), \quad N = \ldots = x^n\psi(v),$$

n being the degree of L, N, and m that of M, and making the above substitution, (α) becomes

$$(v\theta - \psi)(dx/dv) + x^r\phi + x\theta = 0,$$

where $r = m + 2 - n$; that is,

$$\frac{d}{dv}(x^{-(r-1)}) + x^{-(r-1)}\frac{(r-1)\theta}{\psi - v\theta} = \frac{-(r-1)\phi}{\psi - v\theta},$$

the general solution of which is

$$x^{-(r-1)}e^{(r-1)\int\frac{\theta dv}{\psi - v\theta}} = C - (r-1)\int\frac{\phi}{\psi - v\theta}e^{(r-1)\int\frac{\theta dv}{\psi - v\theta}}dv,$$

and the primitive will be given by the substitution of y/x for v after the integration has been performed.

In the Question proposed for solution,

$$\theta = v^3 + 2v^2, \quad \phi = 1 + v, \quad \psi = 2v^3, \quad r = 0,$$

$$\int\frac{\theta dv}{\psi - v\theta} = -\int\left(\frac{dv}{v} + \frac{2dv}{v^2}\right) = +\frac{2}{v} - \log v,$$

and we have

$$xve^{-2/v} = C - \int(1/v^2 + 1/v^3)e^{-2/v}dv$$

$$= C - \tfrac{1}{2}(3 + 2/v)e^{-2/v}.$$

Hence the primitive of (1) is

$$2x + 3y + 4y^2 = C_0 y e^{(2x)/y},$$

where C_0 is an arbitrary constant.

Note.—This substitution can also be successfully employed even when L and N are not homogeneous, provided they can be made so by a linear substitution performed on either x or y, or on both. Thus, for example, the equation (2), when

$$L = ax + \beta y + \gamma, \quad M = Ax + By, \quad N = a'x + \beta' y + \gamma',$$

is reduced to the soluble form

$$L_1 y'_1 + M_1(x_1 y'_1 - y_1) = N_1,$$

where L_1, M_1, N_1 are all homogeneous of the first degree in x_1, y_1 by the substitution $x = x_1 + h$, $y = y_1 + k$, if we choose h and k to satisfy the equations

$$Ah^2 + Bhk + ah + \beta k + \gamma = 0, \quad Ahk + Bk^2 + a'h + \beta' k + \gamma' = 0.$$

Similarly, the solutions of the equations (Forsyth, *Differential Equations*, Vol. I., pp. 40, 41)

$$(a + y + bx + nx^2)y' = y(c + nx), \quad [x(x + y) + a^2]y' = y(x + y) + b^2$$

are more easily obtainable by this means than by any other.

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16012. (Professor NEUBERG.)—Une droite quelconque menée par le pôle O rencontre deux courbes S, S′ aux points M, M′. Les tangentes en ces points rencontrent la perpendiculaire élevée en O sur OM aux points T, T′. Quelle relation faut il établir entre les courbes pour que le conjugué harmonique de O par rapport aux points T, T′ décrive (1) une circonférence de centre O, (2) une droite donnée?

Solution by J. HODGKINSON *and* C. M. ROSS.

Using the notation given by Edwards, *Diff. Calc.*, page 157, the polar equation of the tangent to the first curve is

$$u = U_1 \cos(\theta - \alpha) + U_1' \sin(\theta - \alpha),$$

where α is the vectorial angle of the point of contact. Then the vectorial angle of $T = \alpha + \frac{1}{2}\pi$; therefore $u = U_1'$; *i.e.*, the co-ordinates of T are $(U_1', \alpha + \frac{1}{2}\pi)$, and of T′ are $(U_2', \alpha + \frac{1}{2}\pi)$, α being the same for both points. Then the harmonic conjugate of O with respect to T, T′ is $[\frac{1}{2}(U_1' + U_2'), \alpha + \frac{1}{2}\pi]$.

(1) If the locus of this point is a circle of centre O and radius a, then $\frac{1}{2}(U_1' + U_2') = 1/a$, say; *i.e.*, for any points on the two curves $du_1/d\theta + du_2/d\theta = 2/a$. Integrating, $a(u_1 + u_2) = 2(\theta + \beta)$, where β is an arbitrary constant.

(2) If the locus of this point is a straight line, choose the initial line parallel to this line so that its equation is $up = \sin\theta$. Proceeding as before, we get $p(u_1 + u_2) = 2(\cos\beta - \cos\theta)$.

Note on Factorization by means of Roots.

By ROBT. W. D. CHRISTIE.

The equations are all modular—

(1) **209.**— $N = 9^2 + 2.8^2 = 0 \pmod p$, say,
$= 2^2 + 2.8^2 = 0 \pmod{11}$, since 2, 8 are roots of 11.
Again, $N = 3^2 + 2.10^2 = 0 \pmod{19}$, since 2, 3, 10 are roots of 19.
The factors 11 and 19 are thus indicated.

(2) **1121.**—
$N = 31^2 + 10.4^2 = 31^2 + 10.15^2 = 11^2 + 10.10^2$, indicating 59.19.

(3) **30201.**— $N = 79^2 + 10.86^2 = 10^2 + 10.17^2$, indicating 23.
The other factor
$N = 3487 = 48^2 + 7.13^2 = 4^2 + 7.2^2 = 7^2 + 7.2^2$, indicating 11,
or $2^4 + 7.2^2$, indicating 11,
since $r^4 + r_1.r^2 = 0 \pmod{p_{4m-1}}$.

(4)—Again, since $r^2 + r^{2m+1} = 0 \pmod{4m-1}$,
we have
$r^4 + r_1 r^2 = 0$ where $r_1 = r^{2m+1}$ ..... (1),
$r^4 + r.r_1^2 = 0$ where $r_1 = r^{m+1}$ ..... (2),
$r_1^2 + r_1 r^2 = 0$ where $r_1 = r^{m+1}$ ... . (3),
$r^2 + r.r_1^2 = 0$ where $r_1 = r^m$ ..... (4).

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Again, if $r^2 + r.r_1^2 = 0$,

then $s^2 + s.s_1^2 = 0$ where $rs = 1 = r_1 s_1$ ..... (5),

associated roots, and $s_1^2 + r.s^2 = 0$ from (3) ..... (6).

Similarly $r_1^2 + r_2.r_2^2 = 0$ where $r_1 = r^{3m-3}$ and $r_2 = r^{4m-3}$ ..... (7); and many others.

16163. (Professor COCHEZ.)—On donne trois cercles co-axiaux. Lieu de P tel que les tangentes PA, PB et PC à ces cercles (A, B, C étant les points de contact) vérifient la relation $(PA . PB)/PC^2 = K$.

Solution by HENRY RIDDELL, M.E.

If we have three circles coaxial, then the squares on tangents to two of them from a point on the third are in the ratio of the distances of the centre of the third from the other two centres.

Now, if there are four coaxial circles with centres R, S, T, U, and tangents are drawn from the point P on the circle U to the points A, B, C on the circles R, S, T, we have

$$PA^2/PC^2 = UR/UT \quad \text{and} \quad PB^2/PC^2 = US/UT;$$

therefore $$(PA^2/PC^2)(PB^2/PC^2) = (UR . US)/UT^2$$

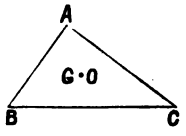
or $$(PA . PB)/PC^2 = [\sqrt{(UR . US)}]/UT.$$

Therefore it is evident that a fourth coaxial circle forms the required locus.

16147. (Professor R. W. GENESE, M.A.)—A given mass is distributed over three given points A, B, C. Prove that the moment of inertia of any distribution about its centroid varies as the power of that point with respect to the circle ABC.

Solution by M. T. NARANIENGAR, M.A.

Let G be the centroid of masses x, y, z at A, B, C whose sum is given, and O the circumcentre of ABC. Then, the moment of inertia of the system about O $= (x+y+z)R^2$, and this should be equal to that about G $+ (x+y+z)OG^2$. Hence the inertia about G



$$= (x+y+z)(R^2 - OG^2) \propto (R^2 - OG^2),$$

or as the power of G with respect to the circle ABC.

16050. (H. L. TRACHTENBERG, B.A.)—The foci of two parabolas and two of the three common tangents other than the line at infinity are given. Construct the third geometrically.

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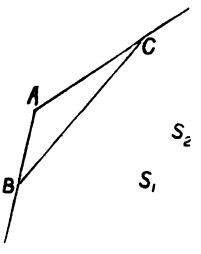
Solution by HENRY RIDDELL, M.E.

Since the image of the focus in the tangent is on the directrix, the two directrices are known. The orthocentre of the triangle formed by three tangents to a parabola lies upon the directrix, and, since there are two directrices, the orthocentre is their common point. Hence the orthocentre and two sides of a triangle are known, and the third side is found by a perfectly simple construction.

The PROPOSER and Mr. W. F. BEARD, M.A., solve the problem as follows :—

Let AB, AC be the given tangents ; S_1, S_2 the foci.

Now, if the third tangent cuts the given tangents in BC, A, B, C, S_1 are concyclic, since the first parabola touches the sides of ABC, and A, B, C, S_2 are concyclic, since the second parabola touches them. Hence A, B, C, S_1, S_2 are concyclic. Thus, if the circle AS_1S_2 be drawn cutting the given tangents in BC, BC is the required tangent.



9266. (Professor MUKHOPÁDHYAY, M.A., F.R.A.S.) — Evaluate

$$f(x, y) = \frac{(x+y)^n}{x^n + y^n} \text{ when } x = y = 0.$$

Solution by SARADAKANTA GANGULI, M.A.

$$f(x, y) = \frac{(x+y)^n}{x^n + y^n} = \frac{[1 + (y/x)]^n}{1 + (y/x)^n}.$$

The value of $f(x, y)$, therefore, depends on (y/x). If, however, $x = y$, $y/x = 1$; therefore $(x, y) = 2^{n-1}$.

16157. (W. F. BEARD, M.A. Suggested by Question 15945.)—T is the pole of a chord PQ of a parabola whose focus is S ; TS, PS, QS meet the circum-circle of the triangle TPQ in T', P', Q'. Prove that the triangles TPQ, T'Q'P' are congruent.

Solutions by (I.) W. J. DOBBS, M.A., *and* M. T. NARANIENGAR, M.A. ; (II.) *by* JAMES BLAIKIE, M.A. ; (III.) *by* F. E. RELTON *and* Professor SANJÁNA, M.A.

(I.) With the figure of Solution II., since $\angle PTS = \angle TQQ' = \angle TPQ'$, PQ' is parallel to TT'. Since $\angle QTS = \angle TPP' = \angle TQP'$, QP' is parallel to TT'. Thus the arcs Q'T', T'P', Q'P' are respectively equal

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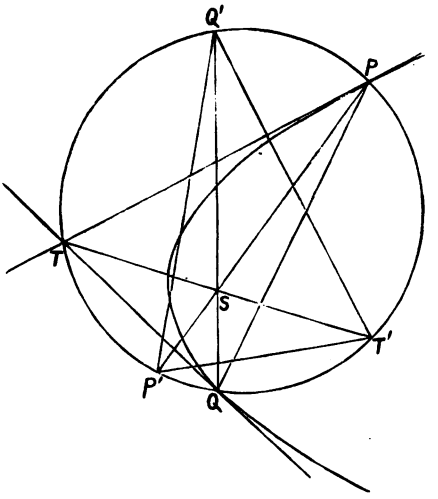
to the arcs PT, TQ, PQ, showing that the triangles TPQ and T'Q'P' are congruent.

(II.) $$\angle P'T'Q' = \angle P'T'T + \angle TT'Q' = \angle P'PT + \angle TQQ'$$
$$= \angle QTS + \angle STP = \angle QTP$$

by properties of circle and parabola;

$$\angle T'P'Q' = \angle T'P'P + \angle PP'Q' = \angle T'TP + \angle PQQ'$$
$$= \angle TQS + \angle PQS = \angle TQP;$$



therefore $$\angle P'Q'T' = \angle QPT;$$

and, since the triangles are similar and in the same circle, they are congruent.

(III.) The circle touching TP at T and passing through Q also goes through the focus. Hence $\angle PTT' = \angle TQS = \angle PP'T'$. Also $\angle PQQ' = \angle PP'Q'$; whence $\angle TQP = \angle T'P'Q'$; so that $TP = T'Q'$, which makes $\angle TT'Q' = \angle T'TP$. The circle touching TQ at T and passing through P also goes through the focus. Hence $\angle QTT' = \angle TT'P'$; so that $\angle QTP = \angle P'T'Q'$. The triangles are therefore equiangular, and, being inscribed in the same circle, are necessarily congruent. As an interesting corollary, QP', TT', and PQ' are all parallel.

The Proposer and Mr. Henry Riddell, M.E., give the following Solution:—

(See Question 15945.) ST is perpendicular to the diameter of the circle TPQ through S, and SP, SQ make equal angles with ST. Thus $ST' = ST$, $SP' = SQ$, $SQ' = SP$, and the triangles TPQ, T'Q'P' are congruent by symmetry as regards the diameter through S of the circle TPQ.

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16168. (Lt.-Col. ALLAN CUNNINGHAM, R.E.) — If ${}^nC_x$ denote the number of combinations of n things taken x together, show that ${}^{p-1}C_x \equiv (-1)^x \pmod p$ when p is prime; and show how far this is true when p is composite. Also show that $\frac{1}{2}({}^{2q}C_q) - 1 \equiv 0 \pmod{q^2}$ when q is a prime > 3; when $q = 3$ the divisor is only q^2.

Solution by the PROPOSER.

By definition,

$${}^{p-1}C_x = \frac{p-1}{1} \cdot \frac{p-2}{2} \cdot \frac{p-3}{3} \cdot \ldots \cdot \frac{p-x}{x} \ (x \text{ factors}) \equiv (-1)^x, \mod p$$
(when p is prime),

because each term reduces to (-1) on casting out the divisor p (provided p be prime). This is not generally true when p is composite; as in that case some of the fractions may reduce to lower terms (less than p itself). But it is true for all (positive integer) numbers (m) whatever that

$${}^{m-1}C_0 = {}^{m-1}C_{m-1} = +1, \quad \text{and} \quad {}^{m-1}C_1 = {}^{m-1}C_{m-2} \equiv -1 \pmod m.$$

The general theorem that ${}^{m-1}C_r \equiv (-1)^r \pmod m$ is also true for all composite numbers (m) so long as every denominator up to the x-th is prime to m: thus

$${}^{34}C_0 = +1, \ {}^{34}C_1 = -1, \ {}^{34}C_2 = +1, \ {}^{34}C_3 = -1, \ {}^{34}C_4 \equiv +1 \pmod{35};$$
but $${}^{34}C_5 \equiv +6, \ {}^{34}C_6 = -6 \pmod{35}.$$

Next, as to ${}^{2q}C_q$ when q is prime:

$$\tfrac{1}{2}{}^{2q}C_q = \tfrac{1}{2}\left(\frac{2q}{1} \cdot \frac{2q-1}{2} \cdot \frac{2q-2}{2} \cdot \ldots \cdot \frac{2q-(q-2)}{q-1} \cdot \frac{2q-(q-1)}{q}\right)$$
(q factors)

$$= \frac{2q-1}{1} \cdot \frac{2q-2}{2} \cdot \frac{2q-3}{3} \cdot \ldots \cdot \frac{2q-(q-2)}{q-2} \cdot \frac{2q-(q-1)}{q-1}$$
($q-1$ factors).

Now take the product of pairs of factors equidistant from the ends, and write (for shortness) $Q = 2q^2$ and $k_x = x(q-x)$. Hereby the general term becomes

$$\frac{2q-x}{x} \cdot \frac{2q-(q-x)}{q-x} = \frac{2q^2 + x(q-x)}{x(q-x)} = \frac{Q}{k_x} + 1.$$

Hence $$\tfrac{1}{2}{}^{2q}C_q = \left(\frac{Q}{k_1} + 1\right)\left(\frac{Q}{k_2} + 1\right) \ldots \left(\frac{Q}{k_m} + 1\right) \quad [m = \tfrac{1}{2}(q-1)];$$

and, since q is prime, every denominator k_x is prime to $\frac{1}{2}Q = q^2$; hence, casting out the modulus q^2, every factor leaves the residue $(+1)$ to mod q^2, and $\quad \frac{1}{2}{}^{2q}C_q \equiv +1 \quad$ and $\quad \frac{1}{2}{}^{2q}C_q - 1 \equiv 0 \pmod{q^2}$,

and this is true for all odd primes q.

Next, multiplying all the factors together, and casting out the modulus $\frac{1}{4}Q^2 = q^4$, all terms containing Q^2 and higher powers of Q vanish, so that

$$\tfrac{1}{2}{}^{2q}C_q \equiv AQ + 1 \pmod{q^4} \quad \text{and} \quad \tfrac{1}{2}{}^{2q}C_q - 1 \equiv AQ \pmod{q^4},$$

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where $A = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_m}$ $[m = \frac{1}{2}(q-1)]$

$$= \frac{1}{(q-1)!}\left\{\frac{(q-1)!}{k_1} + \frac{(q-1)!}{k_2} + \ldots + \frac{(q-1)!}{k_m}\right\};$$

and, by a known theorem, $A \equiv 0 \pmod q$, when q is an odd prime > 3. Hence, finally, $\frac{1}{2}\,{}^{2q}C_q - 1 \equiv 0 \pmod{q^3}$.

Ex.—Writing, for shortness, $N_q = \frac{1}{2}\,{}^{2q}C_q - 1$, we have

$$N_3 = 3^2, \quad N_5 = 5^3, \quad N_7 = 5.7^3, \quad N_{11} = 5.11^3.53,$$
$$N_{13} = 3^2.13^3.263, \quad N_{17} = 17^3.53.4481, \quad N_{19} = 13.19^3.198197.$$

Note that, writing, for shortness, $q = \kappa + 2$,

$${}^{2q}C_q = 2\,\frac{(2q-1)(2q-2)(2q-3)}{q(q-1)^2}\,{}^{2\kappa}C_\kappa,$$

by which formula each ${}^{2q}C_q$ may be obtained from a previously calculated ${}^{2\kappa}C_\kappa$. (q, κ need not be primes.)

16171. (M. V. ARUNACHALAN, M.A. *Suggested by* Question 16083, Professor SANJÁNA, M.A.)—**Find the value of the infinite products**

(1) $(1 + x^4/1^4)(1 + x^4/2^4)(1 + x^4/3^4)$, ...,
(2) $(1 + x^4/1^4)(1 + x^4/3^4)(1 + x^4/5^4)$, ...

Solution by C. M. Ross.

The following is a particular case of Dr. Glaisher's solution of the general case :— $1 + x^2 = (1 - \alpha x)(1 - \alpha^3 x)$ ..... (1),

in which $\alpha = \cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi$, or any other prime 4th root of 1.

Put x^2, $(\frac{1}{2}x)^2$, $(\frac{1}{3}x)^2$, ... successively for x in (1); then

$$\left.\begin{aligned} 1 + x^4 &= (1 - \alpha x^2)(1 - \alpha^3 x^2) \\ 1 + \frac{x^4}{2^4} &= \left(1 - \alpha\frac{x^2}{2^2}\right)\left(1 - \alpha^3\frac{x^2}{2^2}\right) \\ 1 + \frac{x^4}{3^4} &= \left(1 - \alpha\frac{x^2}{3^2}\right)\left(1 - \alpha^3\frac{x^2}{3^2}\right) \\ \ldots \quad & \ldots \quad \ldots \quad \ldots \\ x^4 &= -\alpha^2 x^2 . \alpha^3 x^2 \end{aligned}\right\} \ldots\ldots\ldots\ldots \text{(A)}.$$

Equations (A) may be written

$$\left.\begin{aligned} 1 + x^4 &= \{1 - (\beta x)^2\}\{1 - (\beta^3 x)^2\} \\ 1 + \frac{x^4}{2^4} &= \left\{1 - \left(\frac{\beta x}{2}\right)^2\right\}\left\{1 - \left(\frac{\beta^3 x}{2}\right)^2\right\} \\ \ldots \quad & \ldots \quad \ldots \quad \ldots \quad \ldots \\ x^2 &= (-)^{\frac{1}{2}}\beta x . \beta^3 x \end{aligned}\right\} \ldots\ldots\ldots\ldots \text{(B)},$$

where $\beta = \alpha^{\frac{1}{2}} = \cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi = p_1 + iq_1$ and $p_{n-r} = q_r$;

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whence

$$x^2\left(1+\frac{x^4}{1^4}\right)\left(1+\frac{x^4}{2^4}\right)\ldots = -\frac{1}{\pi^2}\sin\pi\beta x\,\sin\pi\beta^2 x$$

$$= -\frac{1}{\pi^2}\sin\pi x(p_1+iq_1)\sin\pi x(p_3+iq_3)$$

$$= -\frac{1}{\pi^2}\sin\pi x(p_1+iq_1)\sin\pi x(-p_1+iq_1)$$

$$= -\frac{1}{2\pi^2}\{\cos 2\pi p_1 x-\cos 2\pi i q_1 x\}$$

$$= \frac{1}{2\pi^2}\{\cosh 2\pi q_1 x-\cos 2\pi p_1 x\}.$$

Also from these formulæ

$$\left(1+\frac{x^3}{1^4}\right)\left(1+\frac{x^4}{3^4}\right)\left(1+\frac{x^4}{5^4}\right)\ldots = \tfrac{1}{2}\{\cosh\pi q_1 x+\cos\pi p_1 x\}.$$

16158. (V. Ramaswami Aiyar, M.A.)—ABC is a triangle inscribed in a conic, focus S. If the base BC passes through the reflection of the vertex A with respect to the S-directrix, then the pedal circle of the triangle ABC with respect to S has its diameter equal to SA.

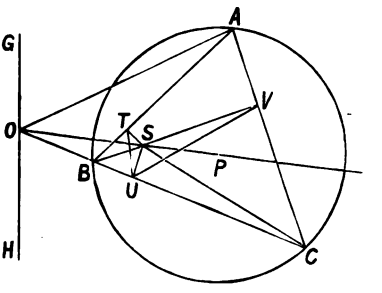
Solution by Henry Riddell, M.E.

If BC passes through the reflection of A in the directrix GH, then, if BC cuts GH in O, we have AO, CO making equal angles with GH. Therefore

SA/AO = SB/BO = SC/CO,

and the circum-circle of ABC is the locus of vertices of triangles having base SO and sides in the ratios named. Hence the centre P lies on OS, and we have

$$PS.PO = PA^2.$$



Therefore S lies on the inverse of BC with respect to P, which is a circle passing through BPC, and

$$\angle BSC = \angle BPC = 2\angle BAC.$$

Then $\angle SBA + \angle SCA = \angle BAC$ and $\angle TBS = \angle SUT$

(T, U, V being feet of perpendiculars on sides), since SUBT is cyclic. Similarly

$$\angle VCS = \angle SUV \quad \text{and} \quad \angle TUV = \angle BAC.$$

Hence the circum-circle of TUV has a diameter equal to that of the circle passing through TAV, that is, equal to SA.

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16090. (H. BATEMAN, B.A.)—A_1, A_2, A_3, A_4 are any four points in a plane. Out of these any point such as A_4 is chosen, and lines are drawn through A_1, A_2, A_3 perpendicular to A_4A_1, A_4A_2, and A_4A_3 respectively, so as to form a new triangle $B_1B_2B_3$. Prove that the ratio of the areas of the triangles $A_1A_2A_3$, $B_1B_2B_3$ is the same whichever point A_4 is chosen.

Solution by the PROPOSER.

The proof of the proposition depends upon the following considerations:—

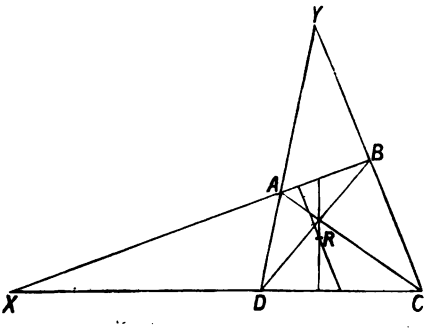
(1) If we project orthogonally the figure formed by a triangle and its orthocentre, it is easy to see that each of the conics having one point, say A_3, as centre and the other three as a conjugate triad has the same eccentricity.

(2) If A_4 is the centre of a conic having $A_1A_2A_3$ as a conjugate triad and a triangle $B_1B_2B_3$ is obtained in the manner described, the ratio of the triangles $A_1A_2A_3$, $B_1B_2B_3$ depends only on the eccentricity of the conic.

If one of your readers can obtain a simpler demonstration, I shall be very glad.

16183. (Professor R. W. GENESE, M.A.)—Inward forces act at the mid-points of the sides of any plane quadrilateral at right angles and proportional to the opposite sides. Prove (1) that they balance, (2) that the resultant of two opposite forces acts along the directrix of the parabola inscribed in the quadrilateral.

Solution by S. T. SHOVELTON, M.A.



(1) Let each force be replaced by its two equivalent parallel forces at the ends of the line at the middle point of which it acts. Then at A there are now two forces, one perpendicular and proportional to BC, and the other perpendicular and proportional to CD. Their resultant is perpendicular and proportional to BD. So also the resultant at C is perpendicular and proportional to BD. These form a couple of moment proportional to $AC \cdot BD \cos\theta$, where θ is the angle between the diagonals. So also the forces at B and D form an equal couple which balances the first, and the system is therefore in equilibrium.

(2) The orthocentres of the four triangles XAD, XBC, YAB, YCD lie

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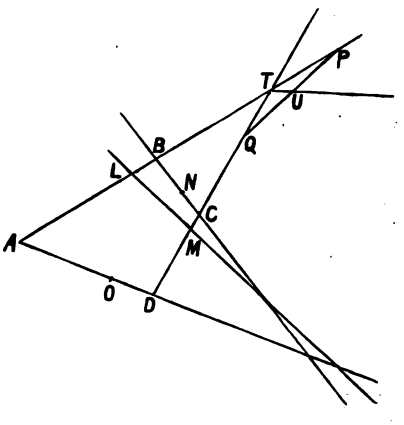
on the directrix of the parabola inscribed in ABCD. Let the lines of action of two opposite forces meet at R and those of the other pair at S. It can easily be proved that R is collinear with the orthocentres of XAD and XBC. Therefore R is on the directrix. Similarly S is on the directrix. But RS must be the line of action of either pair of opposite forces, and therefore the resultant of either pair acts along the directrix.

A more general theorem of which Question 16183 is a special case is discussed by Mr. HENRY RIDDELL, M.E., as follows:—

Let ABCD be a quadrilateral, and let AB, DC be cut in L and M, so that AL/LB = DM/MC. Let also the other pair of sides be cut so that

CN/NB = DO/OA.

Then, if through the four points thus found forces be acting proportional to the opposite sides, and in the directions of the perpendiculars to those opposite sides, the same theorem will hold.



Let the parabola inscribed in the quadrilateral touch the sides A and CD in P and Q respectively; then we know that PT/TA = TQ/QD, by the usual theory of intersecting tangents to a parabola. Also TP/TB = TQ/QC, and we see that TP/TQ = BA/CD. But forces acting at T in the directions and proportional to TP and TQ have their resultant along the diameter TU, and therefore the resultant of the forces through L and M which are proportional and at right angles to these must be at right angles to the axis.

It is evident that the resultants of the opposite pairs through L and M, and O and N, must be equal in magnitude. It only remains to show that they both act along the same line to prove they balance, and, if this line is the directrix, the entire theorem is proved. But the line LM is evidently a tangent to the same parabola, as is also the line ON. The perpendiculars of the triangle formed by any three tangents meet on the directrix. Therefore the lines of forces acting through L and M perpendicular to TM and TL respectively meet on the directrix, and their resultant passes through this point, and, being perpendicular to the axis, lies along the directrix.

16114. (*Suggested in conversation by the late* Professor P. G. TAIT. *Communicated by* J. BLAIKIE, M.A.)—A. and B. play golf, and at each hole it is equally probable that the hole may be won by A., won by B.,

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or halved. Find A.'s chances of winning, halving, or losing the match, if he is (1) 2 up and 3 to play, (2) 2 up and 4 to play, (3) m up and $m+n$ to play.

Solution by H. M. TAYLOR, M.A., F.R.S.

Let A.'s chance at any particular hole of winning, halving, or losing be p, q, r respectively, where $p+q+r=1$. Let

$$(px+q+rx^{-1})^{m+n} = C_{m+n}x^{m+n} + \ldots + C_s x^s + \ldots + C_0 + \ldots + C_{-r}x^{-r} + \ldots + C_{-(m+n)}x^{-(m+n)}.$$

Then, if there are $m+n$ holes to be played, A.'s chances of altering his lead by $+s$, by 0, or by $-t$ are C_s, C_0, and C_{-t} respectively. If, therefore, A. is at present m up, his chance of halving is C_{-m}, his chance of winning is $C_{m+n} + \ldots + C_{-(m-1)}$, and his chance of losing the match is $C_{-(m+1)} + \ldots + C_{-(m+n)}$. In the special cases given, we can put $p=q=r=\frac{1}{3}$, whence the chances of winning, halving, and losing the match will be in case (1) $\frac{33}{27}$, $\frac{3}{27}$, $\frac{1}{27}$, and in case (2) $\frac{96}{81}$, $\frac{19}{81}$, $\frac{6}{81}$ respectively.

Remarks by A. M. NESBITT, M.A.

[As a golfer's "remarks" will not always bear reproduction in print, I hasten to assure your readers that I am not a golfer.]

The value of m has no influence on A.'s chance of winning any specified number of holes in those that still remain to play: it will merely affect the number he has to win or halve in order to halve the match—more than this number will win for him, less will lose. If N/D be his chance of *scoring* r holes, *i.e.* of winning s and halving $2(r-s)$, out of n holes to play, I find the following:—

$$\begin{array}{lllllll} r = 0, & \frac{1}{2}, & 1, & 1\frac{1}{2}, & 2, & 2\frac{1}{2} & \\ \mathrm{N} = 1, & n, & \frac{n(n+1)}{2!}, & \frac{n(n-1)(n+4)}{3!}, & \frac{n(n-1)(n^2+7n-6)}{4!}, & & \\ & & & & & \frac{n(n^2-1)(n-2)(n+12)}{5!}; & \end{array}$$

and, of course, $\mathrm{D} = 3^n$. In the two special cases given we get

	$r=$	0,	$\frac{1}{2}$,	1,	$1\frac{1}{2}$,	2,	$2\frac{1}{2}$,	3,	$3\frac{1}{2}$,	4;
(1)	N =	1,	3,	6,	7,	6,	3,	1,	—,	—; D = 27;
(2)	N =	1,	4,	10,	16,	19,	16,	10,	4,	1; D = 81;

so that in (1) A's chances of losing, halving, and winning are respectively $\frac{1}{27}$, $\frac{3}{27}$, $\frac{23}{27}$, and in (2) they are $\frac{5}{81}$, $\frac{19}{81}$, $\frac{96}{81}$.

Further than $r=2\frac{1}{2}$ I have not gone in the general case; in fact, I would far sooner learn to play golf than attempt such a feat. If the solution of this problem be regarded as a "hole" at that pastime, I am inclined to think that it is more likely to be halved than won by your contributors.

In the case of m up and $m+n$ to play, we must make $r=\frac{1}{2}n$ for a tie, and write $m+n$ instead of n in the above results. I have not done so, as I conceive they are quite unwieldy enough as they stand.

Mr. BLAIKIE's Solution is as follows:—

(1) A. will lose if B. wins all three holes. The probability of this

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is clearly $1/3^3 = \frac{1}{27}$. The match will be halved if B. wins two and halves one hole. As this may be done in three ways, the probability of its happening is $3.1/3^3 = \frac{3}{27}$. Thus A.'s chance of winning the match is $1 - \frac{4}{27} = \frac{23}{27}$.

This may also be shown by considering the different ways A. may win and the chances of each. He may win by winning the first hole—chance $\frac{1}{3}$.

If we write a for a hole won by A., b for one won by B., and 0 for one halved, then A. may win the match in the next two holes by any of the following combinations:—00, $0a$, ba—chance $\frac{2}{3}$. Or he may win in three holes by $0ba$, $0b0$, $b00$, $b0a$, bba—chance $\frac{5}{27}$. His whole chance is therefore $\frac{1}{3} + \frac{2}{3} + \frac{5}{27} = \frac{23}{27}$.

(2) Here B. wins if he gains all four holes or if he gains three and halves the fourth. His chance is therefore $\frac{4}{81}$. The game is halved if either B. wins three holes and A. wins the fourth ($\frac{8}{81}$) or if B. wins two holes and halves two ($\frac{6}{81}$). Total chance of halving $\frac{14}{81}$. Thus A.'s chances of winning is $1 - \frac{4}{81} - \frac{14}{81} = \frac{63}{81} = \frac{23}{27}$. This may be verified as in (1).

(3) I have not been able to establish a general formula for all values of n. The following formulæ hold good when $n = 1, 2, 3$:—

	Ratios of Chances of A.'s		
n	Winning.	Halving.	Losing.
1	$3^{m+1} - (m+2)$	$m+1$	1
2	$3^{m+2} - \frac{1}{2}(m+2)(m+4)$	$\frac{1}{2}(m+2)(m+3)$	$m+3$
3	$3^{m+3} - \frac{1}{6}(m^3 + 15m^2 + 68m + 156)$	$\frac{1}{6}(m+2)(m+3)(m+7)$	$\frac{1}{2}(m+4)(m+5)$

The proofs can be easily established; but the work is long and is not particularly interesting.

16152. (M. V. Arunachalan, M.A.)—Show that

$$[a(b-c)(bc-aa')(a^4-a'^4)]/(a-a') + [b(c-a)(ca-bb')(b^4-b'^4)]/(b-b') + [c(a-b)(ab-cc')(c^4-c'^4)]/(c-c') = (b-c)(c-a)(a-b)(bc-aa')(ca-bb')(ab-cc')\,S/abc$$

where $aa' = bb' = cc'$ and $S = a+b+c+a'+b'+c'$.

Solution by F. W. Reeves, M.A., *and* Lalit Mohon Mukherji, M.A., B.L.

In the given expression put $b = c$, and then, from the given condition $bb' = cc'$, we get $b' = c'$, and the expression vanishes; therefore $b-c$ and similarly $(c-a)$ and $(a-b)$ are factors. Again, put $b = c'$, and

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then $b' = c$, and the given expression again vanishes; therefore $(b'-c)$, $(c'-a)$, and $(a'-b)$ are factors.

Now the given expression is of the seventh degree and we have six factors; therefore the remaining factor is LS, where L is numerical and $S = \Sigma a + \Sigma a'$. Also

$b'-c = -(bc-aa')/b$, $c'-a = -(ca-bb')/c$, and $a'-b = -(ab-cc')/a$; therefore the given expression

$$= -L(b-c)(c-a)(a-b)(bc-aa')(ca-bb')(ab-cc')\,S/abc.$$

Equating coefficients of a^4b^2c on both sides, we find that $L = 1$; therefore $\Sigma[a(b-c)(bc-aa')(a^4-a'^4)]/(a-a')$

$$= (b-c)(c-a)(a-b)(bc-aa')(ca-bb')(ab-cc')\,S/abc.$$

N.B.—The symmetry of this result shows that, with the same conditions,

$\Sigma[a'(b'-c')(b'c'-aa')(a^4-a'^4)]/(a-a')$

$$= (b'-c')(c'-a')(a'-b')(b'c'-aa')(c'a'-bb')(a'b'-cc')\,S/a'b'c'.$$

16088. (Professor NANSON.)—Find the condition that two given conics may intersect a given line in the same two points.

Solution by CONSTANCE I. MARKS, B.A.

Take rectangular axes and the given line as Ox.

Let the equations to the given conics be respectively

$$a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots(1)$$

and

$$a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots(2).$$

These cut $y = 0$ where

$$a_1x^2 + 2g_1x + c_1 = 0, \quad a_2x^2 + 2g_2x + c_2 = 0 \quad \ldots\ldots\ldots\ldots(3, 4).$$

The required condition is the condition that (3) and (4) may have the same roots, and it follows that we must have

$$a_1/a_2 = g_1/g_2 = c_1/c_2.$$

15999. (D. M. Y. SOMMERVILLE, M.A., D.Sc.)—Find the forms of the equation of a conic which is its own reciprocal.

Discussion of the Question by $\theta.\beta$.

Consider the reciprocal of a conic S with respect to another, S'; suppose that one of the intersections of S, S' is P. Then the reciprocal of P with respect to S' is PT, the line touching S' at P; by hypothesis PT touches S, and must therefore touch S at P. It follows that S, S'

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touch at their common points; and they must therefore have double contact.

It is easily verified that, if we write

$$S' = ax^2 + by^2 + cz^2, \quad S = kS' + (lx + my + nz)^2,$$

then the only further condition is $l^2/a + m^2/b + n^2/c + 2k = 0$. In particular, with $a = b = c = 1$, we find

$$S = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2) - 2(lx + my + nz)^2,$$

agreeing with the results given by the Proposer in Vol. XI., New Series.

16169. (R. W. D. CHRISTIE.)—When p is any prime of form $4m-1$ and r_1, r_2 are two different prime roots, prove that $r_1^2 + r_1 r_2^2 \equiv 0 \pmod{p}$. Give a few other similar equations and show their utility in factorization. *E.g.*, $10^2 + 10.3^2 \equiv 0 \pmod{19}$; also

$$2^2 + 2.13^2 = 3^2 + 3.15^2 = 13^2 + 13.14^2 \ldots.$$

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

The proposition requires some limitation. For $r_1^2 + r_1 r_2^2 \equiv 0 \pmod{p}$ is equivalent to $r_1 + r_2^2 \equiv 0$. Now, if r_2 be any primitive root of p, then

$$r_1^{\frac{1}{2}(p-1)} \equiv -1 \quad \text{and} \quad r_1^{(p-1)} \equiv +1 \pmod{p},$$

and no lower exponents will yield these (∓ 1) residues. Next, taking a number r_1, such that

$$r_1 = p - r_2^2, \quad \text{or} \quad \equiv p - r_2^2, \quad \text{or} \quad \equiv -r_2^2 \pmod{p};$$

then $r_1^{\frac{1}{2}(p-1)} \equiv (\mathrm{I})^{\frac{1}{2}(p-1)}$, $r_1^{p-1} \equiv (\mathrm{I})(+1) \equiv -1 \pmod{p}$,

and no exponent less than $\frac{1}{2}(p-1)$ will suffice to yield this (-1) residue. Hence r_1 is also a primitive root. Thus to every primitive root (r_2) belongs one, and only one, primitive root satisfying the congruence

$$r_1 + r_2^2 \equiv 0.$$

Ex.—The pairings of the primitive roots for $p = 19$ are

$$2^2 + 15 \equiv 3^2 + 10 \equiv 10^2 + 14 \equiv 13^2 + 2 \equiv 14^2 + 13 \equiv 15^2 + 3 \equiv 0.$$

A general extension of above to residuacity of any hyper-even order n is as follows:—

Let $n = 2^\nu$, $p = 2n\omega + 1$, ω an odd number, and let r_2 be any number such that $(r_2/p)_{2n} = -1$; then the number r_1, given by

$$r_1 + r_2^2 \equiv 0 \pmod{p},$$

is also a residue of the same order n, *i.e.*, $(r_1/p)_n = -1$.

Ex.—$p = 281 = 8.5.7 + 1$; $n = 4$; $(2/p)^8 = -1$; $277 + 2^2 = p$; therefore $(277/p)_8 = -1$.

The following Solution is due to the PROPOSER:—

From previous work by the Proposer we have, when $p = 4m-1$, $r^2 + r_1 \equiv 0 \pmod{v}$, where r and r_1 are both prime roots of p; thus,

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$r_1^2 + r_1 r^c = 0 \pmod{p}$, and here we may substitute $R_1 = pk + r_1$ and $R = pl + r$ for r_1 and r respectively, where k and l are perfectly arbitrary. The theorem is adapted to factorization thus: let

$$N = 22987^2 + 10 \cdot 23017^2;$$

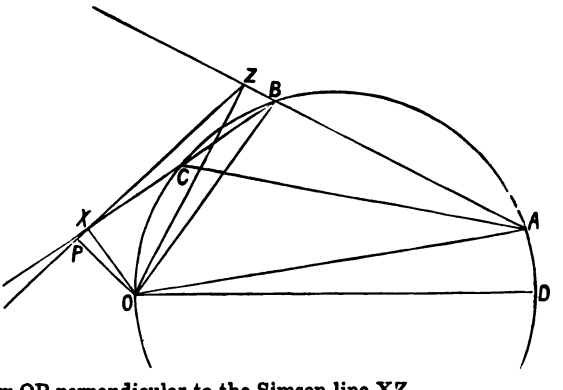
then we may detect the factor 23 from the fact that $23017 = 23k + 17$ and $22987 = 23l + 10$, where 10 and 17 are both prime roots of 23. The principle is of wide application, and other root equations may be used to get a second factor, *vide* previous volumes, especially Questions 14593, 14668, 14616, in Vol. LXXV., p. 116, &c., and Question 15168, Vol. IV., New Series, &c.

A Theorem in Analytical Geometry.

By W. GALLATLY, M.A.

O *being the pole of the circle* $r = 2a\cos\theta$, *and* α, β, γ *being the vectorial angles of* A, B, C, *to prove the Simson line of* O *is*

$$r\cos[\theta - (\alpha + \beta + \gamma)] = 2a\cos\alpha\cos\beta\cos\gamma.$$



Draw OP perpendicular to the Simson line XZ,

$$\angle ZOA = 90° - \angle ZAO = 90° - \angle BDO = \beta,$$

$$\angle POZ = 90° - \angle XZO = 90° - \angle XBO = 90° - \angle CDO = \gamma;$$

therefore $\angle POD = \alpha + \beta + \gamma$.

Also $OP = OZ\cos\gamma = OA\cos\beta\cos\gamma = 2a\cos\alpha\cos\beta\cos\gamma$;

therefore, &c.

16176. (Professor LAUVERNAY.)—Construire la courbe

$$\rho^2 - 4\rho\cos\tfrac{1}{3}\omega + 1 = 0.$$

Solution by HENRY RIDDELL, M.E.

The curve $\rho^2 - 4\rho\cos\frac{1}{3}\omega + 1 = 0$ can be constructed from the related

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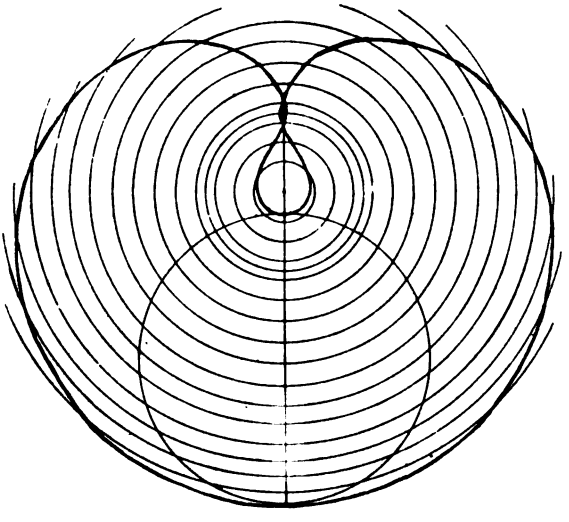
curve $\rho^2-4\rho\cos\omega+1=0$ in the following way. Construct the circle represented by the latter curve, which has a radius $\sqrt{3}$, and origin at



a distance 2 from the centre. With the origin as centre draw a series of circles cutting the circle already drawn, and upon these circles set off arcs, measured from the line joining the origin to the centre of the original circle, and equal to three times the intercept between this axis and the circumference. These are points on the curve, and, if joined, give a curve resembling that drawn. The proof is obvious. The series of such curves can be made to include the circle and ellipse. In the same way the general curve $\rho^2-4a\rho\cos\omega/n+b=0$ may be constructed.

16145. (W. Austin Sleigh, B.A.)—In Question 15909 prove that, if ξ be centre of ellipse, I be in-centre, and I_1 ex-centre, corresponding to A, of the triangle ABC, (1) foot of directrix, K, is at distance $s/2e$ from ξ; (2) BI meets AX at distance $e(s-c)$ from X, and BI_1 meets it at distance $es^2/(3c-b-a)$ from ξ; (3) length of semi-diameter conjugate to ξB is $\sqrt{(r_1r_2)}$; (4) $I_1B : I_1C = \sqrt{r_2} : \sqrt{r_3}$; (5) the parallel straight line to BI_1 through ξ meets BC at distance $(BL =)\, s/2$ from B; (6) $4\xi T.CL = es^2$, T being intersection of BI_1 and AX; (7) ratio of CL to distance of foot of B ordinate from ξ is that of the eccentricity.

Solution by the Proposer.

(1) We have $\xi X.\xi K = (\frac{1}{2}s)^2$, $\frac{1}{2}(AX)\,\xi K = \frac{1}{4}s^2$, $\frac{1}{2}es.\xi K = \frac{1}{4}s^2$;

therefore $\xi K = s/2e$.

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(2) We have $XG = eXB = e(s-c)$.

Again, $\xi G \cdot \xi T = \xi X^2$, $(\xi X - XG)\xi T = \xi X^2$, $(\frac{1}{2}AX - XG)\xi T = \xi X^2$,

$$[\tfrac{1}{2}es - e(s-c)]\,\xi T = \tfrac{1}{4}e^2s^2, \quad e(c-\tfrac{1}{2}s)\,\xi T = \tfrac{1}{4}e^2s^2;$$

whence $$\xi T = es^2/(3c-a-b).$$

(3) Semi-diameter conjugate to ξB

$$= \sqrt{(XB \cdot AB)} = \sqrt{[s(s-c)]} = \Delta/\{\sqrt{[(s-a)(s-b)]}\} = \sqrt{(r_1r_2)}.$$

(4) Since I_1B, I_1C are tangents to the ellipse, they are in the ratio of the semi-conjugate diameters; therefore

$$I_1B : I_1C = \sqrt{(r_1r_2)} : \sqrt{(r_2r_3)} = \sqrt{r_1} : \sqrt{r_3}.$$

(5) Since BI_1 is tangent at B, the straight line through ξ parallel to BI_1 is the direction of the conjugate diameter to ξB, and therefore meets the focal chord XB at L; hence $BL = \frac{1}{2}s$.

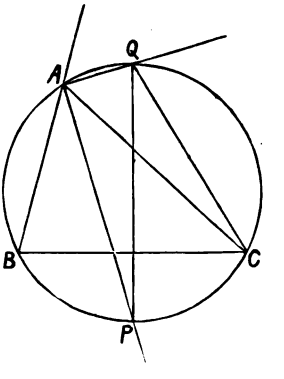
(6) Since $BL = \frac{1}{2}s$, $CL = \frac{1}{4}(3a-b-c)$. Combining this with (2), we have $$4\xi T \cdot CL = es^2.$$

(7) We have $\xi N \cdot \xi T = (\frac{1}{2}s)^2$; therefore $e\xi N \cdot \xi T = \xi T \cdot CL$, where $$CL/\xi N = e.$$

16200. (Professor SANJÁNA, M.A.)—The great circle bisecting perpendicularly the side BC of the spherical triangle ABC meets the great circles bisecting internally and externally the angle A in the points D, E. Prove that D and E lie on the small circle circumscribing ABC.

Solution by HENRY RIDDELL, R.E.

Draw a plane through ABC, and consider the traces of the various planes containing the great circles. Since these planes all pass through one common point, the centre of the sphere, and this point is equally distant from A, B, and C in the new plane drawn, it is evident that the traces of the two planes containing the bisecting great circles themselves bisect the angle at A in this plane as internal and external bisectors. In the same way the trace of the great circle plane bisecting BC of the original triangle bisects and is at right angles to BC in the new plane.



The points of intersection of these traces lie on the lines common to pairs of planes, and are on the circle circumscribing ABC, but this circle is the small circle round ABC, and these points of intersection become the points of intersection of great circles.

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16048. (A. M. NESBITT, M.A.)—If three points $(am^2, 2am)$ on the parabola $y^2 = 4ax$ be determined by the roots of the cubic

$$m^3 - pm^2 + qm - r = 0,$$

the equation of the circle circumscribing the triangle formed by tangents at these points will be

$$x^2 + y^2 - ax(1+q) - ay(p-r) + a^2q = 0.$$

Solutions (I.) *by* Professor NANSON; (II.) *by* W. F. BEARD, M.A. (III.) *by* C. M. ROSS *and* F. W. REEVES, M.A.

(I.) The tangent at m is $am^2 - my + x = 0$ .....(1). Hence, if (x, y) is a vertex of the triangle in question, it follows that (1), considered as a quadratic in m, has two roots in common with the given cubic, and therefore also in common with

$$(ap-y)m^2 - (aq-x)m + ar = 0;$$

therefore $ap - y = ra^2/x, \quad aq - x = rya/x;$

therefore $apq + aqx - x^2 - y^2 = r\,a^2y/x + ray = a^2q - ax + ray,$

and this is the required equation.

(II.) Let m_1, m_2, m_3 be the roots of

$$m^3 - pm^2 + qm - r = 0 \text{ .....(i.).}$$

Any conic circumscribing the triangle formed by the tangents at am_1^2, $2am_1$, ... is given by

$$\lambda(x - m_2y + am_2^2)(x - m_3y + am_3^2) + \mu(\quad)(\quad) + \nu(\quad)(\quad) = 0 \text{...(ii.).}$$

If this is a circle, we have $\Sigma\lambda(1 - m_2m_3) = 0$, $\Sigma\lambda(m_2 + m_3) = 0$. Thus

$$\lambda/[(m_2 - m_3)(1 + m_1^2)] = \mu/[(\quad)(\quad)] = \nu/[(\quad)(\quad)].$$

Thus the coefficient of $x^2 + y^2$ in (ii.) is

$$\Sigma m_1^2(m_2 - m_3) + (\Sigma m_2 - m_3) = -\Pi(m_2 - m_3).$$

The coefficient of x

$$= a\Sigma(m_2 - m_3)(m_2^2 + m_3^2) + a\Sigma m_1^2(m_2 - m_3)(m_2^2 + m_3^2)$$
$$= a\Pi(m_2 - m_3) + a\Pi(m_2 - m_3)\Sigma m_2m_3;$$

coefficient of y

$$= -a\Sigma m_2m_3(m_2^2 - m_3^2)(1 + m_1^2)$$
$$= -a\Sigma m_2m_3(m_2^2 - m_3^2) - am_1m_2m_3\Sigma m_1(m_2^2 - m_3^2$$
$$= a\Pi(m_2 - m_3)\Sigma m_1 - am_1m_2m_3\Pi(m_2 - m_3).$$

The last term

$$= a^2\Sigma m_2^2m_3^2(m_2 - m_3) + a^2m_1^2m_2^2m_3^2\Sigma(m_2 - m_3)$$
$$= -a^2\Pi(m_2 - m_3\Sigma m_2m_3);$$

therefore the equation of the required circle is

$$x^2 + y^2 - ax(1 + \Sigma m_2m_3) - ay(\Sigma m_1 - m_1m_2m_3) + a^2\Sigma m_2m_3 = 0;$$

or, from (1), $\quad x^2 + y^2 - ax(1+q) - ay(p \quad r) + a^2q = 0.$

(III.) The tangents to the parabola are $m_r y = x + am_r^2$ $(r = 1, 2, 3)$. (1) and (2) intersect in P $[am_1m_2, a(m_1 + m_2)]$, (2) and (3 in

$$Q\,[am_2m_3,\ a(m_2 + m_3)$$

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(3) and (1) in R $[am_2m_1, a(m_3+m_1)]$. Now, m_1, m_2, m_3 are the roots of the cubic $m^3-pm^2+qm-r=0$; therefore $\Sigma m_1 = p$, $\Sigma m_1m_2 = q$, $m_1m_2m_3 = r$. If the equation of the circum-circle PQR is

$$x^2+y^2+2gx+2fy+c=0,$$
$$a^2[m_1^2m_2^2+(m_1+m_2)^2]+2agm_1m_2+2af(m_1+m_2)+c=0,$$
$$a^2[m_2^2m_3^2+(m_2+m_3)^2]+2agm_1m_2+2af(m_1+m_2)+c=0,$$
$$a^2[m_3^2m_1^2+(m_3+m_1)^2]+2agm_1m_2+2af(m_1+m_2)+c=0.$$

From these equations $g=-\frac{1}{2}(1+q)a$, $f=-\frac{1}{2}(p-r)$, $c=a^2q$. Therefore the circle is $x^2+y^2-a(1+q)x-a(p-r)y+a^2q=0$.

Addendum to Note, p. 90, supra.

By R. W. D. CHRISTIE.

$r^2+n : r^l = 0 \pmod p$, where $n = r^k \pmod p$ and $k+l = 2m+1$, $p = 4m-1$.

Thus
$$7^2+1.7^7 = 7^2+1.6 = 0 \pmod{11},$$
$$7^2+2.7^4 = 7^2+2.3 = 0 \pmod{11},$$
$$7^2+3.7^3 = 7^2+3.2 = 0 \pmod{11},$$
$$7^2+4.7^1 = 7^2+4.7 = 0 \pmod{11},$$
$$7^2+5.7^5 = 7^2+5.10 = 0 \pmod{11},$$
$$7^2+6.7^0 = 7^2+6.1 = 0 \pmod{11},$$
$$7^2+7.7^6 = 7^2+7.4 = 0 \pmod{11},$$
$$7^2+8.7^8 = 7^2+8.9 = 0 \pmod{11},$$
$$7^2+9.7^9 = 7^2+9.8 = 0 \pmod{11},$$
$$7^2+10.7^2 = 7^2+10.5 \equiv 0 \pmod{11},$$

and we may say $N = 11a+n$; thus

$$7^2+5.10 = 7^2+16.10 = 0 \pmod{11} = 7^2+27.10 = 0 \pmod{11},$$

and more generally $N = pa+n$ where a is any integer and n any integer except p, and we may also say $7^2+3.2 = 0 = 4^2+3.2, \ldots$.

Remarks by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Some further explanation would be interesting if the process is meant as an aid to factorization (of large numbers).

(i.) What is meant by the term "roots"? Does it mean "primitive roots"? If so, there seems to be some mistake in Ex. 2, N = 1121.

(ii.) How are the trial primes p selected? And how are three "roots" to be found when p is large; or, if random numbers be taken, how are they to be recognized as "roots" when p is large?

(iii.) How would the process be applied to $N = (2^{67}-1)$, which is known to have two large prime factors?

Note.—With reference to the above, Mr. CHRISTIE writes as follows:—

(*a*) When the first root is found all others are readily obtained by

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use of a multiplier as explained in Proposer's Question in a previous volume.

(*b*) From (3) in "Note," p. 90, *supra*, we have $r_1^2 + r.r^2$ where $r_1 = r^{m+1}$. Here $r_1 = 10^6 = 11$. Hence

$$11^2 + 10.10^2 = 19.59 = 1121.$$

(*c*) The Proposer recommends the use of some such equations (there is a large number) as

$$[2n(m-l) \pm m]^2 \pm 2nl^2 \equiv [2n(m-l) \mp m]^2 \pm 2n(2m-l)^2 \ldots (1, 2),$$

$$[n(m-l) \pm m]^2 \pm nl^2 \equiv [n(m-l) \mp m]^2 \pm n(2m-l)^2 \ldots\ldots (3, 4)$$

to factorize integers of form $2^k - 1$: *e.g.*, $k = 11$,
$N = 45^2 + 22 \cdot 1^2$ by square root $= 43^2 + 22 \cdot 3^2$ ($n = 11$, $l = -1$, $m = +1$).

6372. (Rev. J. L. KITCHIN, M.A.)—Find the sum of the following series, n and k being integers :—

$$1 + k(n-1) + \frac{k(k-1)}{1.2} \frac{(n-1)n}{1.2} + \frac{k(k-1)(k-2)}{1.2.3} \frac{(n-1)(n)(n+1)}{1.2.3} + \ldots .$$

Solution by CONSTANCE I. MARKS, B.A.

Note.—The Proposer appears to intend n and k to be *positive* as well as integral.

Assuming this, we have, when $x > 1$, $(1+x)^k$ and $(1-1/x)^{-(n-1)}$ both capable of expansion by the binomial theorem ; therefore

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \ldots + x^k$$

and

$$\left(1 - \frac{1}{x}\right)^{-(n-1)} = 1 + (n-1)\frac{1}{x} + \frac{(n-1)n}{2!}\frac{1}{x^2} + \frac{(n-1)n(n+1)}{3!}\frac{1}{x^3} + \ldots,$$

an infinite series.

Now, in the product of $(1+x)^k$ and $(1-1/x)^{-(n-1)}$ the constant term is

$$1 + k(n-1) + \frac{k(k-1)}{1.2} \frac{(n-1)n}{1.2} + \frac{k(k-1)(k-2)}{1.2.3} \frac{(n-1)n(n+1)}{1.2.3}$$

$$+ \ldots \text{ to } k+1 \text{ terms};$$

that is, the series whose sum is required. Now $(1+x)^k (1-1/x)^{-(n-1)}$ may be written

$$\frac{(1+x)^k}{\left(1-\frac{1}{x}\right)^{n-1}} = \frac{(x+1)^k}{\left(1-\frac{1}{x}\right)^{n-1}}$$

$$= \frac{x^k + kx^{k-1} + \frac{k(k-1)}{2!} x^{k-2} + \ldots + 1}{1 - (n-1)\frac{1}{x} + \frac{(n-1)(n-2)}{2!}\frac{1}{x^2} - \ldots + (-1)^{n-1}\frac{1}{x^{n-1}}},$$

where both series terminate.

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Hence, if the division be performed, the sum required is the constant term in the quotient, *i.e.*, the $(k+1)$-th term.

The work in the general case is perfectly straightforward, but long; the truth of the results may, however, at once be verified by giving small particular values to k and n: *e.g.*, $k=2$, $n=3$ and $k=2$, $n=2$.

In the first instance the sum of three terms of the given series is 8. Also the third or $(k+1)$-th term of the quotient resulting from

$$(x^2+2x+1)/[1-2(1/x)+1/x^2],$$

namely, $x^2+4x+8+\ldots$, is 8.

Again, in the second instance, the sum of three terms of the given series is $1+2.1+1=4$, and the third or $(k+1)$-th term of the quotient resulting from $(x^2+2x+1)/(1-1/x)$, namely, $x^2+3x+4+\ldots$, is 4.

16161. (V. Daniel, B.Sc.) — The perpendiculars of A'B'C' are rotated in the same direction through an angle ϕ about the vertices, forming a triangle (modulus of similarity $1/\lambda \equiv 2\sin\phi$) capable of being revolved round some point O through an angle θ, so as to lie inscribed to Δ' (A on B'C', ...). Show that

$$4\sin\theta = \sqrt{(4\lambda^2-1)} \pm \sqrt{(4\lambda^{-2}-1)}.$$

Solution by Professor Sanjána, M.A.

It is seen that AB, BC, CA, after revolution of Δ, make respectively with C'A', A'B', B'C' the angles $\theta+\phi+A-\frac{1}{2}\pi$, $\theta+\phi+B-\frac{1}{2}\pi$, $\theta+\phi+C-\frac{1}{2}\pi$. Hence

$$A'C/BC = [\sin(\theta+\phi+A-\tfrac{1}{2}\pi+B)]/\sin A$$

or $$A'C = 2k\sin\phi\cos(\theta+\phi-C)$$

where $$k = a'/\sin A' = b'/\sin B' = c'/\sin C';$$

so also $$B'C = -2k\sin\phi\cos(\theta+\phi+C).$$

Thus $$A'B' = 2k\sin\phi.2\sin(\theta+\phi)\sin C,$$

i.e., $$4\sin\phi\sin(\theta+\phi) = 1.$$

Solving this equation, we readily get

$$(4\sin\theta\cos\phi\sin\phi-1)^2 = \sin^4\phi(16-16\sin^2\theta)$$

or $$(x\cos\phi\sin\phi-1)^2 = \sin^4\phi(16-x^2);$$

whence $$x\sin\phi = \cos\phi \pm \sqrt{(16\sin^4\phi-\sin^2\phi)}$$

or $$x = \cot\phi \pm \sqrt{(16\sin^2\phi-1)},$$

which is equivalent to the result given.

The Proposer discusses the Question as follows:—

It is obvious geometrically that, if the perpendiculars of ABC be similarly rotated through an angle ϕ' subject to the condition

$$4\sin\phi\sin\phi' = 1,$$

we shall produce a triangle Δ'', circumscribed to Δ, which is Δ' revolved through the angle $(-\theta)$ and have merely to revolve it back to co-

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incidence with Δ', carrying Δ with it. That there are two centres of revolution, and that therefore θ has two values, is plain: for Δ'' may be produced by positive or negative rotation of the perpendiculars of Δ through the angle ϕ'.

Now $(-\theta)$ = angle between corresponding sides of Δ' and Δ''
= positive rotation of A''B'' relative to A'B'
$= [(\frac{1}{2}\pi - B') + \phi] + [-(\frac{1}{2}\pi - B) \pm \phi'] = \phi \pm \phi'$:

therefore $\sin\theta = \sin\phi' \cos\phi \pm \cos\phi' \sin\phi$

therefore $4\sin\theta = \cot\phi \pm \cot\phi'$, since $(2\sin\phi)(2\sin\phi') \equiv \lambda\lambda^{-1} = 1$;

therefore $4\sin\theta = \sqrt{(4\lambda^2 - 1)} \pm \sqrt{(4\lambda^{-2} - 1)}$.

It is interesting to deduce the same fact from the general case of two such mutually rotatable triangles dividing one another's sides in ratios determined by the six quantities k, l, m, k', l', m'. Referring to Question 15273, Vol. VI., New Series, writing s for $\sin\theta$, equations (1) and (2) are:

$$s(\cot\omega - \cot A) = l(1 + m') - l'(1 + m) \quad \ldots\ldots (1),$$

$$s(\cot\omega' - \cot A') = m(1 + l') - m'(1 + l) \quad \ldots\ldots (2);$$

therefore $s(\cot\omega + \cot\omega') = \Sigma k - \Sigma k'$

and $s(\cot\omega - \cot\omega') = \Sigma lm' - \Sigma l'm$;

therefore $s(\cot A + \cot A') = k - k' \quad \ldots\ldots (3)$

and $s(\cot C - \cot B') = lm' - l'm \quad \ldots\ldots (4)$.

By subtraction of (1) and (2),

$$s(\cot\omega' - \cot\omega + \cot A - \cot A') = (1 + l + l')(m - m') - (1 + m + m')(l - l').$$

Now, if $\omega = \omega'$, we obtain

$$(1 + l + l')(m - m') - (1 + m + m')(l - l') = s(\cot A - \cot A').$$

of this be due to direct similarity of Δ and Δ', the triangles are geometrically related as in this Question; consequently

$$(1 + l + l')(m - m') - (1 + m + m')(l - l') = 0$$

and its analogues. Therefore

$$\frac{k - k'}{1 + k + k'} = \frac{l - l'}{1 + l + l'} = \frac{m - m'}{1 + m + m'} \equiv p, \quad \text{say};$$

therefore $$\frac{1 + 2k}{1 + 2k'} = \frac{1 + 2l}{1 + 2l'} = \frac{1 + 2m}{1 + 2m'} \equiv \frac{1 + p}{1 - p};$$

therefore $$\left(\frac{1 + p}{1 - p}\right)^2 = \frac{\Sigma(1 + 2k)(1 + 2l)}{\Sigma(1 + 2k')(1 + 2l')} = \frac{4\lambda^2 - 1}{4\lambda'^2 - 1};$$

therefore $$v = \frac{\sqrt{(4\lambda^2 - 1)} - \sqrt{(4\lambda'^2 - 1)}}{\sqrt{(4\lambda^2 - 1)} + \sqrt{(4\lambda'^2 - 1)}} \quad \text{or} \quad \frac{\sqrt{(4\lambda^2 - 1)} + \sqrt{(4\lambda'^2 - 1)}}{\sqrt{(4\lambda^2 - 1)} - \sqrt{(4\lambda'^2 - 1)}}.$$

Again, the sum of the products of (3) and its analogues in pairs gives

$$4s^2 \Sigma(\cot A \cot B) = \Sigma[(k - k')(l - l')];$$

i.e., $4s^2 = p\Sigma[(k - k')(1 + l + l')] = p(\lambda^2 - \lambda'^2)$, since $\Sigma(lm' - l'm) = 0$;

therefore $16s^2 = [(4\lambda^2 - 1) - (4\lambda'^2 - 1)]$ (p_1 or p_2);

therefore $4s = \sqrt{(4\lambda^2 - 1)} \pm \sqrt{(4\lambda'^2 - 1)}$;

i.e., $4\sin\theta = \sqrt{(4\lambda^2 - 1)} \pm \sqrt{(4\lambda^{-2} - 1)}$, since $\lambda^2\lambda'^2 = 1$.

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16028. ("SOLIDUS.")—Find, without the aid of tables, to as great a degree of approximation as possible, the real root of the equation $x^{x+1} = (x+1)^x$. The calculations are to be capable of being carried out by a good computer in half an hour.

Solution by CONSTANCE I. MARKS, B.A.

$x^{x+1} = (x+1)^x$; therefore $\left(\frac{x+1}{x}\right)^x \frac{1}{x} - 1 = 0$ or

$$\left(1+\frac{1}{x}\right)^x \frac{1}{x} - 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If $x = 2$, (1) gives for its left-hand member $\frac{9}{8} - 1 = \frac{1}{8}$, which is fairly close. Again, $x = \frac{5}{2}$ gives on left of (1)

$$\left(\frac{7}{5}\right)^{\frac{5}{2}} \frac{2}{5} - 1 \quad \text{or} \quad \frac{49}{25} \times \sqrt{\frac{7}{5}} \times \frac{2}{5} - 1;$$

that is,
$$\frac{49}{25} \times \frac{\overset{13}{\cancel{25}}}{\underset{11}{\cancel{22}}} \times \frac{2}{5} - 1 = \frac{1274 - 1375}{1375},$$

which is negative; therefore x lies somewhere between 2 and $2\frac{1}{2}$.

For the calculations of roots see end of Solution.

Let $x = 2\frac{1}{4}$; then left of (1) is

$$\left(\frac{13}{9}\right)^{2+\frac{1}{4}} \times \frac{4}{9} - 1 = \frac{169}{81} \times \frac{11}{10} \times \frac{4}{9} - 1 = \frac{7436 - 7290}{7290} = \frac{146}{7290}$$

$$= \frac{73}{3645} = \frac{1}{50} \text{ approximately.}$$

Let $x = 2\frac{1}{3}$; then left of (1) becomes

$$\left(\frac{10}{7}\right)^{2+\frac{1}{3}} \times \frac{3}{7} - 1 = \frac{100}{49} \times \sqrt[3]{\frac{10}{7}} \times \frac{3}{7} - 1$$

$$= \frac{100}{49} \times \frac{22}{19} \times \frac{3}{7} - 1 = \frac{6600 - 6517}{6517}$$

$$= \frac{83}{6517} = \frac{1}{80} = \cdot 0125 \text{ approximately;}$$

therefore the real root of the given equation is $2\frac{1}{3}$ or 2·33 nearly.

The actual calculations can be made by a good computer in half an hour.

Calculation for $\sqrt{7}$.—	2	7	(2·64
		4	
	46	300	
		276	
	524	·2400	
		2096	
Calculation for $\sqrt{5}$.—	2	5	(2·23
		4	
	42	100	
		84	
	443	1600	
		1329	

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Calculation for $\sqrt{13}$.—

	3	13	3·60
		9	
	66	400	
		396	
	720	400	

Therefore $\sqrt[4]{\frac{13}{9}} = \sqrt{(1{\cdot}20)} = \frac{11}{10}$ nearly.

Calculation for $\sqrt[3]{10}$.—

	2	10	(2·15
		8	
1200		2000	
60			
1			
1261		1261	
132300		·739000	
3150			
25			
1354758		677375	

Calculation for $\sqrt[3]{7}$.—

	1	7	(1·91
		1	
300		6000	
270			
81			
651		5859	
108300		141000,	&c.

9725. (A. E. Thomas.)—Solve the equations

$$6(x^2+yz)^2+4x(y^3+z^3) = a^4+5x^4 \quad \ldots\ldots(1),$$
$$6(y^2+zx)^2+4y(z^3+x^3) = b^4+5y^4 \quad \ldots\ldots(2),$$
$$6(z^2+xy)^2+4z(x^3+y^3) = c^4+5z^4 \quad \ldots\ldots(3).$$

Solution by S. Narayanan, B.A., L.T.

The equations may be written as

$$x^4+12x^2yz+4x(y^3+z^3)+6y^2z^2 = a^4 \quad \ldots\ldots(A),$$
$$y^4+12y^2xz+4y(z^3+x^3)+6z^2x^2 = b^4 \quad \ldots\ldots(B),$$
$$z^4+12z^2xy+4z(x^3+y^3)+6x^2y^2 = c^4 \quad \ldots\ldots(C);$$

A + B + C gives $(x+y+z)^4 = a^4+b^4+c^4$;

therefore $x+y+z = (a^4+b^4+c^4)^{\frac{1}{4}} \quad \ldots\ldots (D).$

Also $x+\omega y+\omega^2 z = (a^4+\omega b^4+\omega^2 c^4)^{\frac{1}{4}}$,

$x+\omega^2 y+\omega z = (a^4+\omega^2 b^4+\omega c^4)^{\frac{1}{4}}$,

where ω, ω^2 are the imaginary cube roots of unity.

Hence $x = \frac{1}{3}\{(a^4+b^4+c^4)^{\frac{1}{4}}+(a^4+\omega b^4+\omega^2 c^4)^{\frac{1}{4}}+(a^4+\omega^2 b^4+\omega c^4)^{\frac{1}{4}}\}$,

and similar results for y and z by multiplying the above results by ω, ω^2, ...

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